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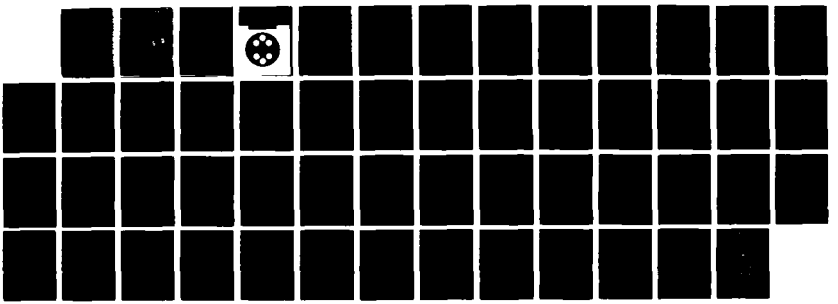

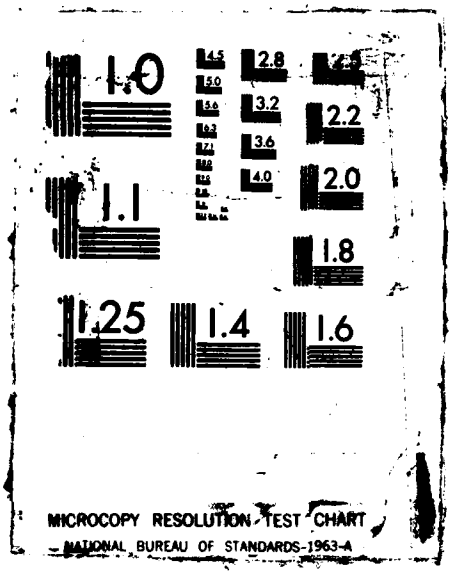
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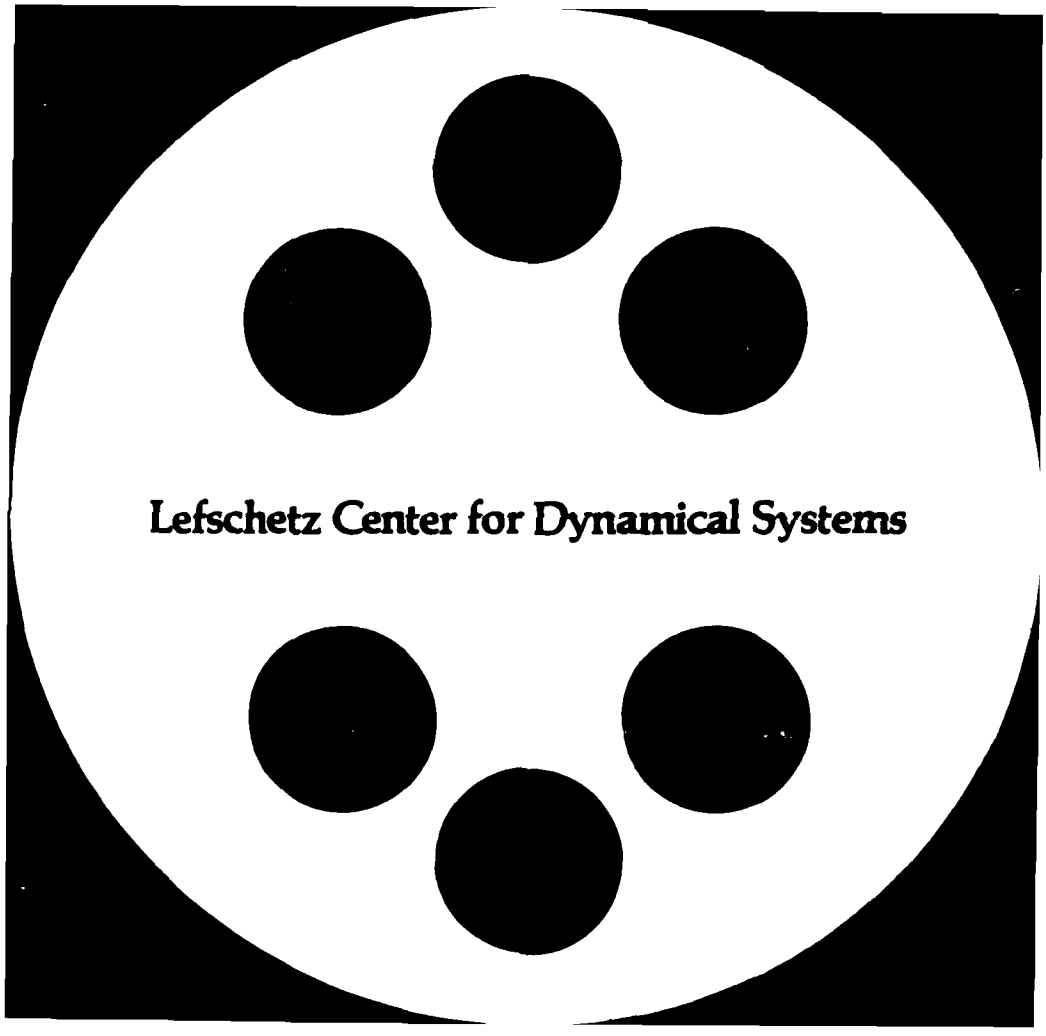
20. Abstract.

Singular control problems with diffusion or Wiener process systems have been occuring with increasing frequency as models of a wide variety of applications; e.g., storage, inventory, finite fuel, consumption and investment, limits of impulsive control problems, etc. Here, the increment of the control force is not of the usual form u(t)dt, but is the differential of a non-decreasing and suitably adapted process. The models used (Wiener or diffusion processes) are only approximations in some sense to some 'physical' process - perhaps a 'wideband' noise driven system or a suitably scaled discrete parameter process. The optimal controls for these 'physical' processes are usually nearly impossible to obtain. Thus, it is of considerable interest to know whether the optimal (or δ-optimal control for the diffusion model is 'nearly' optimum when applied to the physical problem, when compared to the optimal or δ-optimal control for the latter problem. This is true, under broad conditions. The discounted and average cost per unit time problems are treated. The main methods are those of weak convergence theory. But the usual weak convergence analysis via the Skorohod topology on $D[0,\infty)$ is not appropriate here, due to the nature of the singular controls. A combination of the Skorohod and 'pseudopath' topology is adapted to our singular control problem to give the required results.


Lefschetz Center for Dynamical Systems

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August 1986 LCDS #86-43

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NEARLY OPTIMAL SINGULAR CONTROLS FOR WIDEBAND NOISE DRIVEN SYSTEMS

by

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August 1986

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Abstract

Singular control problems with diffusion or Wiener process systems have been occuring with increasing frequency as models of a wide variety of applications; e.g., storage, inventory, finite fuel, consumption and investment, limits of impulsive control problems, etc. Here, the increment of the control force is not of the usual form $u(t)dt$, but is the differential of a non-decreasing and suitably adapted process. The models used (Wiener or diffusion processes) are only approximations in some sense to some 'physical' process - perhaps a 'wideband' noise driven system or a suitably scaled discrete parameter process. The optimal controls for these 'physical' processes are usually nearly impossible to obtain. Thus, it is of considerable interest to know whether the optimal (or δ-optimal control for the diffusion model is 'nearly' optimum when applied to the physical problem, when compared to the optimal or δ-optimal control for the latter problem. This is true, under broad conditions. The discounted and average cost per unit time problems are treated. The main methods are those of weak convergence theory. But the usual weak convergence analysis via the Skorohod topology on $D[0,\infty)$ is not appropriate here, due to the nature of the singular controls. A combination of the Skorohod and 'pseudopath' topology is adapted to our singular control problem to give the required results.

Key Words: singular stochastic control, approximately optimal control, weak convergence, pseudopath topology, control of wideband noisy systems, modelling of physical systems by singular control processes.

1. Introduction

Let $Y_i(\cdot)$, $i = 0,1$, be non-decreasing processes with $Y_i(0) = 0$ and which are non-anticipative with respect to a Wiener process $w(\cdot)$. Define $x(\cdot)$, $Y(\cdot)$ and $Z(\cdot)$ by $x(0) = x$, $Z(0) = 0$, $Y(\cdot) = Y_0(\cdot) - Y_1(\cdot)$ and (which defines $Z(\cdot)$)

$$dx = b(x)\,dt + \sigma(x)\,dw + dY_0 - dY_1 = dZ + dY \quad . \tag{1.1}$$

We assume that there is a $\bar{B} \in (0,\infty)$ such that we are obliged to keep $x(t) \in [0,\bar{B}]$. Unless otherwise mentioned, we always assume that the $Y(\cdot)$ process is such that $x(t) \in [0,\bar{B}]$. The process (1.1) has been widely used as a model of storage and dam processes, both with and without control [1] - [5]. The $Y_1(\cdot)$ might denote the 'withdrawal' process, whereby actual use is made of the system's contents. $Y_0(\cdot)$ might simply denote a process which is used solely as a modelling device to guarantee that $x(t) \geq 0$ (in that case, $dY_0(t) = 0$ if $x(t) \neq 0$). See, e.g. [4]. The process $Z(\cdot)$ might denote the difference between the 'natural' inputs and 'natural' demand. See, in particular, the discussion in [4] on this point. Many other interpretations are possible and are discussed in the references.

Let $k_0 > k_1 > 0$ and let $k(\cdot)$ be a bounded continuous function. Define the two types of costs (E_x denotes the expectation under initial condition $x(0) = x$)

$$V_0(x,Y_0,Y_1) = E_x \int_0^\infty e^{-\beta t} \left[k_0 dY_0(t) - k_1 dY_1(t) + k(x(t))\,dt\right], \tag{1.2}$$

$$V(x) = \inf_{Y_0,\,Y_1} V_0(x,Y_0,Y_1)$$

and

$$\gamma_0(x,Y_0,Y_1) = \overline{\lim_T}\, E_x \int_0^T \left[k_0 dY_0(t) - k_1 dY_1(t) + k(x(t))\,dt\right]/T \tag{1.3}$$

$$\gamma_0(x) = \inf_{Y_0,Y_1} \gamma_0(x,Y_0,Y_1) \quad .$$

In (2.1), the inf is over all non-anticipative $Y_i(\cdot)$ such that $x(t) \in [0,\bar{B}]$. The class over which the inf is taken in (1.3) will be described in Section 7.

Reference [1] gives an elegant presentation of the optimal control problem (1.1), (1.2), and of the properties of the associated Bellman equation. Most of the other current literature seems to concern the case where $Z(\cdot)$ is a Wiener process (perhaps with drift). Since it is highly unlikely that there are problems in the applications which are perfectly modelled by (1.1) or where $Z(\cdot)$ is actually a Wiener process, one must look at the model in the sense that it approximates in some way an actual physical problem. Generally, the model (1.1) would be simpler than the 'physical' process which it approximates. In this sense, it is an attractive object to use for purposes of calculating a control for use on the actual physical process. But the 'quality' of such a control when used on the physical process is far from clear. E.g., how good is it in comparison with the optimal control for the physical process.

Models such as (1.1) often arise as limits of suitably interpolated discrete parameter processes. Consider one type: For each $\epsilon > 0$, let (i = 0,1) $Y_i^\epsilon(\cdot)$ be non-decreasing processes, piecewise constant on the intervals $[n\epsilon, n\epsilon+\epsilon)$, and define $\delta Y_i^\epsilon(n) \equiv Y_i^\epsilon(n\epsilon+\epsilon) - Y_i^\epsilon(n\epsilon)$. For appropriate functions F and G, define $\{X_n^\epsilon, Z_n^\epsilon\}$ by $X_0^\epsilon = x$, $Z_0^\epsilon = 0$, and

$$
\begin{aligned}
X_{n+1}^\epsilon &= X_n^\epsilon + (Z_{n+1}^\epsilon - Z_n^\epsilon) + \delta Y_0^\epsilon(n\epsilon) - \delta Y_1^\epsilon(n\epsilon) \\
Z_{n+1}^\epsilon - Z_n^\epsilon &= \epsilon G(X_n^\epsilon, \xi_n^\epsilon) + \sqrt{\epsilon}\, F(X_n^\epsilon, \xi_n^\epsilon) \qquad (1.4) \\
EF(x, \xi_n^\epsilon) &\equiv 0 ,
\end{aligned}
$$

where $\{\xi_n^\epsilon\}$ is some correlated sequence of random variable. We say that the controls $\delta Y_i^\epsilon(n\epsilon)$ in (1.4) at time n are admissible if they depend on the 'full information' $\{X_i^\epsilon, i \leq n, Y_j^\epsilon(\epsilon i), j = 0,1, i \leq n, \xi_j^\epsilon, j < n\}$ available at time n. Define the interpolated processes $X^\epsilon(t) = X_n^\epsilon$ and $Z^\epsilon(t) = Z_n^\epsilon$ on $[n\epsilon, n\epsilon+\epsilon)$ and the costs

(using admissible controls)

$$V_0^\epsilon(x,\ Y_0^\epsilon,\ Y_1^\epsilon) = E_x^\epsilon \int_0^\infty e^{-\beta t}\left[k_0 dY_0^\epsilon(t) - k_1 dY_1^\epsilon(t) + k(X^\epsilon(t))\,dt\right]$$

(1.5)

$$V^\epsilon(x) = \inf_{Y_0^\epsilon, Y_1^\epsilon} V_0^\epsilon(x,\ Y_0^\epsilon,\ Y_1^\epsilon)\ .$$

In general, we know virtually nothing about the optimal or δ-optimal policies for (1.4), (1.5). Suppose that, for reasonable $Y_i^\epsilon(\cdot)$, the set $\{X^\epsilon(\cdot),\ Y_0^\epsilon(\cdot),\ Y_1^\epsilon(\cdot)\}$ converges weakly in some sense to a solution of (1.1). It is of considerable interest to know just how good (compared with the optimal controls for (1.4)) are the optimal (or δ-optimal) policies which we obtain for (1.1), (1.2), when suitably adapted to and applied to the system (1.4), (1.5). Consider the following example.

Let $\bar{Y}_i(\cdot)$, i=0,1, denote the optimal or (for some given $\delta > 0$) δ-optimal controls for (1.1), (1.2). Frequently [1], [4], they are of the barrier form: there are $0 \leq L^* < U^* < \infty$ such that $\bar{Y}_0(\cdot)$ is used only to keep $x(\cdot)$ from 'falling below' L^* and $\bar{Y}_1(\cdot)$ is used only to keep $x(\cdot)$ from going above U^*. The policy $\bar{Y}_i(\cdot)$ adapted to (1.4) (call it $\bar{Y}_i^\epsilon(\cdot)$) is a policy which returns $X^\epsilon(\cdot)$ to L^* or to U^* immediately if it ever drops below L^* or exceeds U^*.

From the point of view of optimal control, one wants to show that the costs $V^\epsilon(x)$ and $V^\epsilon(x,\ \bar{Y}_0^\epsilon,\ \bar{Y}_1^\epsilon)$ are close for small ϵ, whether or not the optimal controls for (1.1), (1.2) are of the barrier policy type. It is such a result which justifies the use of the limit model (1.1) to get policies which might be used in applications. Similar comments apply to the cost function (1.3), and also to the continuous parameter analogs of (1.4). This is the class of problems dealt with here. The basic tools are those of weak convergence theory. The work here extends that in [6], [7]. The results in these references cannot be used here.

Owing to the generality of the assumptions on the physical system, there is in general nothing known about the properties of the optimal $Y_i^\epsilon(\cdot)$. Because of this, we cannot (in general) prove tightness or weak convergence of $\{X^\epsilon(\cdot), Y_0^\epsilon(\cdot), Y_1^\epsilon(\cdot)\}$ in the Skorohod topology on $D^3[0,\infty)$. E.g., the jumps in the Y_0^ϵ, Y_1^ϵ might be quite 'wild'--relative to the Skorohod topology. A weaker topology must be used, and it is described in Section 2. In Section 3, we obtain a weak convergence result for a continuous time model, and we show how to extend the results to the discrete time case in Section 4. Section 5 contains some auxiliary results which are needed later. The optimal control problem for the discounted cost case is dealt with in Section 6, where it is shown that the suitably adapted policy for the limit is indeed 'nearly' optimal for the actual physical process, under appropriate and reasonable assumptions. Section 7 concerns the average cost per unit time problem. Here, owing to the natural requirement of stationarity, we impose a Markov structure on the problem.

The basic methods work just as well for many non-scalar models--and several extensions to such models are discussed in Section 8. Applications to queueing and manufacturing networks require a somewhat special development, which will be published separately. There are extensions of the results to cases where the dynamical terms are not smooth or the noise is state dependent. One would then adapt the weak convergence technique and assumptions of [8, Chapters 5.3, 5.5 and 5.8] to the problem here. The methods are quite similar, but the assumptions are more complicated. Our discussion is confined to certain singular control problems. But it is also possible to work with controls which have both a singular and an 'absolutely continuous' component. One simply combines the methods developed here -- with those used in [6] for the non-singular case. The extensions should be straightforward. Possible extensions

in another direction concern the problem of [9], where the singular control problem appears as a limit of impulsive control problems: Let δ_ϵ denote the fixed cost per impulse and (for some $\bar{k} > 0$) $\bar{k}|Y^\epsilon(t) - Y^\epsilon(t^-)|$ the variable cost, for an impulse at time t. Then we can let $\delta_\epsilon \to 0$ as the noise bandwidth goes to ∞, to get an approximation theorem for small fixed impulsive cost and wide bandwidth simultaneously. The method is a simple adaptation of the scheme of this paper.

2. The Pseudopath Topology

In this section, we discuss the topology on $D[0,\infty)$ (replacing the Skorohod topology) which will allow us to obtain the desired weak convergence results.

Let the physical system be modelled by

$$X^\epsilon(t) = Z^\epsilon(t) + Y^\epsilon(t)$$

where $X^\epsilon(0) = x$, $Z^\epsilon(0) = 0$, $Y^\epsilon(\cdot) = Y_0^\epsilon(\cdot) - Y_1^\epsilon(\cdot)$ and define $\bar{X}^\epsilon(\cdot) = \left(X^\epsilon(\cdot), Y_0^\epsilon(\cdot), Y_1^\epsilon(\cdot)\, Z^\epsilon(\cdot)\right)$. The $\bar{X}^\epsilon(\cdot)$ can be viewed either as a continuous parameter interpolation of a discrete parameter system as discussed in Section 1, or it might be an actual physical model for a continuous parameter system. The $Y_i^\epsilon(\cdot)$ can be taken to be either left or right continuous, but we suppose that they (and $X^\epsilon(\cdot)$ and $Z^\epsilon(\cdot)$) are right continuous.

We suppose that the $\bar{X}^\epsilon(\cdot)$ take values in $D^4[0,\infty)$, the space of R^4-valued functions which are right continuous and have left hand limits. [We could let the $Y_i^\epsilon(\cdot)$ take values in a space of measures or of distribution functions, but this is actually not more helpful.] The appropriate topology for our purposes on $D^4[0,\infty)$ is what Meyer [10] and Meyer and Zheng [11] call the *pseudopath topology*. For completeness, we state some definitions and results from [11] which will be needed in the sequel. The results are stated for a scalar valued process, but the natural extensions for the R^r-valued case should be obvious and are used below.

Let $y(\cdot) \in D[0,\infty)$ and define the measure $\lambda(\cdot)$ on the Borel subsets of $[0,\infty)$ by $\lambda(dt) = e^{-t}dt$. Let $\bar{\bar{P}}$ denote the compact space of probability measures (with the weak topology) on the *compactified* space $[0,\infty] \times \bar{R}$, where $\bar{R}$ is the closure of the real line. The *pseudo-path* of $y(\cdot)$ is defined to be the

probability measure on the Borel subsets of $[0,\infty] \times \bar{R}$ which is the image of $\lambda(\cdot)$ under the map $t \longrightarrow (t,y(t))$ of $[0,\infty]$ into $[0,\infty] \times \bar{R}$ (i.e., it is a point in $\bar{\bar{P}}$). Let ψ denote the map which takes $y(\cdot)$ into its pseudo-path, the corresponding point in $\bar{\bar{P}}$. If we write $P = \psi(y(\cdot))$, then the pseudopath P is the measure defined by (A and B are Borel sets in $[0,\infty]$ and $\bar{R}$, resp.) $P(A \times B) = \int_A e^{-t} I_{\{y(t) \in B\}} dt$.

ψ is 1:1 on $D[0,\infty)$, since it identifies all paths which are equal a.e. (Lebesque measure). The topology which $\bar{\bar{P}}$ induces on $D[0,\infty)$ via ψ is called the *pseudo-path topology*. The associated σ-algebra on $D[0,\infty)$ is the same as one gets with the Skorohod topology. In fact ([11], Lemma 1 and comment after its proof), *the pseudo-path topology* on $D[0,\infty)$ is *the topology of convergence in measure.* The same result holds if R and $D[0,\infty)$ are replaced by R^r and $D^r[0,\infty)$, where ψ then maps points $y(\cdot) \in D^r[0,\infty)$ into a measure on the Borel subsets of $[0,\infty] \times \bar{R}^r$. Let $\bar{\bar{P}}_r$ denote the space of probability measures on the Borel subsets of $[0,\infty] \times \bar{R}^r$.

The process $\bar{X}^\epsilon(\cdot)$ induces a measure (which we denote by $\bar{P}_\epsilon$) on $\bar{\bar{P}}_4$ via the pseudo-path mapping ψ. The set $\{\bar{P}_\epsilon\}$ is obviously tight since $\bar{\bar{P}}_4$ is compact. If $\bar{P}$ is a limit measure of any weakly convergent subsequence, then for the convergence to be useful we need at least that $\bar{P}$ be supported by $\psi(D^4[0,\infty))$, since then the limit $\bar{P}$ would correspond to some process $\bar{X}(\cdot) = \left(X(\cdot), Y_0(\cdot), Y_1(\cdot), Z(\cdot)\right)$ with paths in $D^4[0,\infty)$, via the mapping ψ. A convenient criterion for this is given by Meyer and Zheng [11], and will now be described.

Let τ denote a finite partition $\{t_i, i \leq n\} : 0 = t_0 < t_1 < \ldots < t_n = \infty$. Let $U(\cdot)$ denote a process with paths in $D[0,\infty)$ and adapted to a non-decreasing sequence of σ-algebras $\{\bar{\mathcal{F}}_t\}$, and with $E|U(t)| < \infty$ for each $t < \infty$. For convenience in comparing with [11], let $U(t) = 0$ for large t. Define the variations

$$\text{(2.1)} \qquad \begin{aligned} \mathrm{Var}_{\tau}(U) &= \sum_{i \le n} E\left| E_{\mathcal{F}_{t_i}} U(t_{i+1}) - U(t_i) \right| \\ \mathrm{Var}(U) &= \sup_{\tau} \mathrm{Var}_{\tau}(U) . \end{aligned}$$

If $\mathrm{Var}(U) < \infty$, then $U(\cdot)$ is said to be a *quasimartingale.*

For $u < v$, let $N^{u,v}(U)$ denote the number of upcrossings of $U(\cdot)$ on $[0,\infty)$ between the levels u and v. If $U(\cdot)$ is a quasimartingale, then

$$\text{(2.2)} \qquad EN^{uv}(U) \le \frac{|u| + \mathrm{Var}(U)}{v - u} ,$$

an extension of the usual result for martingales ([11], Lemma 3). The main result is ([11], Theorem 4).

Theorem 2.1. *For each* $n = 1,2, \ldots$, *let* P_n *be a probability law on the Borel* subsets of* $D[0,\infty)$ *with the associated process* $U_n(\cdot)$ *being a quasimartingale with* $\sup_n \mathrm{Var}(U_n) < \infty$. *Then there is a subsequence* P_{n_k} *which converges weakly on* $D[0,\infty)$ (*with the pseudo-path topology*) *to a law* P, *and the associated process* $U(\cdot)$ *is a quasimartingale. [Alternatively, let* $\bar{P}_n$ *be the measure induced on* $\bar{\bar{P}}$ *by the map* ψ *acting on* $U_n(\cdot)$. *Then* $\{\bar{P}_n\}$ *is tight on* $\bar{\bar{P}}$ *and there is a weakly convergent subsequence* $\{\bar{P}_{n_k}\}$ *with limit denoted by* $\bar{P}$. $\bar{P}$ *is supported on* $D[0,\infty)$ *and the associated process is a quasimartingale.*]

Combining this with the previous results, we have:

Theorem 2.2. *Assume the conditions and terminology of Theorem* 2.1 *and let* $h(\cdot)$ *be any bounded real valued function on* $D[0,\infty)$ *which is continuous* (w.p.1 *with respect to* P) *when the topology of convergence in measure is used on* $D[0,\infty)$. *Then there is a subsequence* (n_k) *such that* $Eh(U_{n_k}(\cdot)) \to Eh(U(\cdot))$. *Also* ([11], *Theorem* 5) *there is a*

*The Skorohod topology and the pseudopath topology generate the same σ-algebra on $D[0,\infty)$.

further subsequence $\{m_k\} \subset \{n_k\}$ *and a set* I *of full measure (depending on* P*) such that the finite dimensional distributions of* $\{U_m(t), t \in I\}$ *converge to those of* $\{U(t), t \in I\}$. *Let* $f(\cdot)$ *be bounded and continuous on* $[0,\infty)$. *Then* ([11], *Theorem* 6) *the function* $(t_1, ..., t_q) \to Ef(U_{n_k}(t_1), ..., U_{n_k}(t_q))$ *converges in measure to the function* $(t_1, ..., t_q) \to Ef(U(t_1), ..., U(t_q))$.

3. The Quasimartingale Property and Weak Convergence of $\bar{X}^\epsilon(\cdot) = (X^\epsilon(\cdot), Z^\epsilon(\cdot), Y_0^\epsilon(\cdot), Y_1^\epsilon(\cdot))$

A continuous parameter case will be done in this section. The discrete parameter case requires only a few modifications and is discussed in the next section. For concreteness, we use a specific model which is of a widely used form [8], [12], [13], for representing wide band width noise driven systems (with or without the driving $Y_i^\epsilon(\cdot)$ process). The techniques are usable for a much broader class of systems--just as for the case where $Y_i^\epsilon(\cdot) \equiv 0$ dealt with in [6] or the various continuous parameter models in [8]. The model to be used is

$$dX^\epsilon = G(X^\epsilon, \xi^\epsilon)dt + F(X^\epsilon, \xi^\epsilon)dt/\epsilon + dY^\epsilon(t) , \tag{3.1}$$

where $\xi^\epsilon(t) = \xi(t/\epsilon^2)$ and $\xi(\cdot)$ is a right continuous random process.

Let E_t^ϵ denote the expectation conditioned on $\{X^\epsilon(s), \xi^\epsilon(s), Y_0^\epsilon(s), Y_1^\epsilon(s), s \leq t\}$, and E_t the expectation conditioned on $\{\xi(s), s \leq t\}$. Define

$$Z^\epsilon(t) \equiv \int_0^t G(X^\epsilon(s), \xi^\epsilon(s))\,ds + \frac{1}{\epsilon}\int_0^t F(X^\epsilon(s), \xi^\epsilon(s))\,ds .$$

We will use the following assumptions. Various extensions (vector case, discontinuous dynamics, state dependent noise) are possible, as discussed in the introduction, via the appropriate extension of the methods in [8] for these cases to the problem at hand.

A3.1. $G(\cdot,\cdot)$, $F(\cdot,\cdot)$, *and* $F_x(\cdot,\cdot)$ *are bounded continuous functions and the latter two are continuous in* x- *uniformly in* ξ.

A3.2. For each scalar x, $EF(x, \xi(s)) \equiv 0$ *and* $\xi(\cdot)$ *is right continuous and sufficiently mixing such that there is a* $K < \infty$ *for which for each* $T < \infty$

$$\sup_{x,t\leq T} \left| \int_t^T E_t\, g(x, \xi(s))\,ds \right| \leq K , \tag{3.2}$$

where $g(\cdot,\cdot)$ *represents either* $F(\cdot,\cdot)$ *or* $F_x(\cdot,\cdot)$.

Theorem 3.1. *Assume* (A3.1) *and* (A3.2) *and let* $\sup_{\epsilon}\left[E Y_0^{\epsilon}(t) + E\, Y_1^{\epsilon}(t)\right] < \infty$ *for each* t. *Then (with the addition of a process whose maximum value goes to zero as* $\epsilon \to 0$*),* $\{X^{\epsilon}(\cdot), Y_0^{\epsilon}(\cdot), Y_1^{\epsilon}(\cdot), Z^{\epsilon}(\cdot)\}$ *are quasimartingales with uniformly (in* ϵ*) bounded variation on each bounded time interval. [We need not assume that* $X^{\epsilon}(t) \in [0,\bar{B}]$ *in this theorem.]*

Proof. Since the mean variations of the $Y_i^{\epsilon}(\cdot)$ are bounded, they are obviously quasimartingales with uniformly (in ϵ) bounded variation on each interval [0,t]. Thus, we need only work with the $Z^{\epsilon}(\cdot)$. We will use the so-called perturbed test function method [8], [12], [14] but adapted to our present needs. For some arbitrary--but large--T, define the process $Z_1^{\epsilon}(\cdot)$ for $t \leq T$, by

$$Z_1^{\epsilon}(t) = \frac{1}{\epsilon}\int_t^T E_t^{\epsilon} F(X^{\epsilon}(t),\ \xi^{\epsilon}(s))\,ds = \epsilon \int_{t/\epsilon^2}^{T/\epsilon^2} E_t^{\epsilon} F(X^{\epsilon}(t),\ \xi(s))\,ds$$

The change of variable $s/\epsilon^2 \longrightarrow s$ will be used frequently, in the averaging and bounding in the sequel, when working with integrals such as $Z_1^{\epsilon}(\cdot)$. By (A3.2)

$$\operatorname*{Sup}_{t \leq T} |Z_1^{\epsilon}(t)| = 0(\epsilon)\,.$$

We will show that the function defined by $f^{\epsilon}(t) = Z^{\epsilon}(t) + Z_1^{\epsilon}(t)$ is a quasimartingale with uniformly (in ϵ) bounded variation on each interval [0,T]. The calculations will be done in a slightly indirect way so that they can be re-used later. Let $f(\cdot)$ denote a 'test' function with bounded and continuous derivatives up to order three, and define $f^{\epsilon}(t) = f(Z^{\epsilon}(t)) + f_1^{\epsilon}(t)$, where $f_1^{\epsilon}(\cdot)$ is the 'perturbation' defined by

$$f_1^{\epsilon}(t) = \frac{1}{\epsilon}\int_t^T f_x(Z^{\epsilon}(t))E_t^{\epsilon}\ F(X^{\epsilon}(t),\ \xi^{\epsilon}(s))ds$$

$$= \epsilon \int_{t/\epsilon^2}^{T/\epsilon^2} f_x(Z^{\epsilon}(t))E_t^{\epsilon}\ F(X^{\epsilon}(t),\ \xi(s))ds.$$

Write $Y^{\epsilon}_{i\,c}(\cdot)$, $Y^{\epsilon}_{i\,d}(\cdot)$ and $Y^{\epsilon}_{c}(\cdot)$, $Y^{\epsilon}_{d}(\cdot)$ for the continuous and jump components of $Y^{\epsilon}_{i}(\cdot)$ and $Y^{\epsilon}(\cdot)$ resp. By our convention on the right continuity of the $Y^{\epsilon}_{i}(\cdot)$, we use $dY^{\epsilon}_{i\,d}(u) = Y^{\epsilon}_{i\,d}(u) - Y^{\epsilon}_{i\,d}(u^{-})$. By integrating the derivative of $f(Z^{\epsilon}(\cdot))$,

$$E^{\epsilon}_{t}\, f(Z^{\epsilon}(t+\Delta)) - f(Z^{\epsilon}(t)) =$$

$$(3.3)\qquad E^{\epsilon}_{t} \int_{t}^{t+\Delta} f_{z}(Z^{\epsilon}(u)) \left[G(X^{\epsilon}(u), \xi^{\epsilon}(u)) + \frac{F(X^{\epsilon}(u), \xi^{\epsilon}(u))}{\epsilon} \right] du.$$

Similarly, by evaluating $[E^{\epsilon}_{u} f^{\epsilon}_{1}(u+\Delta) - f^{\epsilon}_{1}(u)]/\Delta$ and letting $\Delta \to 0$, we get

$$(3.4)\qquad E^{\epsilon}_{t}\, f^{\epsilon}_{1}(t+\Delta) - f^{\epsilon}_{1}(t) = -\frac{1}{\epsilon} \int_{t}^{t+\Delta} f_{z}(Z^{\epsilon}(u))\, E^{\epsilon}_{t}\, F(X^{\epsilon}(u), \xi^{\epsilon}(u))\, du$$

$$+ \int_{t}^{t+\Delta} du\, \frac{1}{\epsilon}\, E^{\epsilon}_{t} \int_{u}^{T} ds\, f_{zz}(Z^{\epsilon}(u))\, E^{\epsilon}_{u}\, \frac{F(X^{\epsilon}(u), \xi^{\epsilon}(s))}{\epsilon} \cdot$$

$$\cdot \left[\frac{F(X^{\epsilon}(u), \xi^{\epsilon}(u))}{\epsilon} + G(X^{\epsilon}(u), \xi^{\epsilon}(u)) \right]$$

$$+ E^{\epsilon}_{t} \int_{t}^{t+\Delta} dY^{\epsilon}_{c}(u)\, \frac{1}{\epsilon} \int_{u}^{T} f_{z}(Z^{\epsilon}(u))\, E^{\epsilon}_{u}\, F_{x}(X^{\epsilon}(u), \xi^{\epsilon}(s))\, ds$$

$$+ \int_{t}^{t+\Delta} E^{\epsilon}_{t} \left[\frac{F(X^{\epsilon}(u), \xi^{\epsilon}(u))}{\epsilon} + G(X^{\epsilon}(u), \xi^{\epsilon}(u)) \right]_{x} du \cdot$$

$$\cdot \int_{u}^{T} E^{\epsilon}_{u} f(Z^{\epsilon}(u))\, \frac{F(X^{\epsilon}(u), \xi^{\epsilon}(s))\, ds}{\epsilon} +$$

$$+ \sum_{t<u\le t+\Delta} \frac{1}{\epsilon}\, E^{\epsilon}_{t} \int_{u}^{T} f_{z}(Z^{\epsilon}(u))\, E^{\epsilon}_{u} \left[F(X^{\epsilon}(u^{-}) + dY^{\epsilon}_{d}(u), \xi^{\epsilon}(s)) \right.$$

$$\left. - F(X^{\epsilon}(u^{-}), \xi^{\epsilon}(s)) \right] ds$$

Recall that $dY(u) = Y(u) - Y(u^-)$.

By a change of scale $s/\epsilon^2 \longrightarrow s$ and the use of (A3.1) and (A3.2), the second and fourth terms on the r.h.s. of (3.4) are seen to be $O(\Delta)$. By a similar scale change, the third term is seen to be $O(\epsilon)E_t^\epsilon\,(Y_c^\epsilon(t+s) - Y_c^\epsilon(t))$. The first term of (3.4) is the negative of the '$1/\epsilon$' term in (3.3). For the evaluation of the last term in (3.4), first use the law of the mean to rewrite it as

$$(3.5)\qquad \sum_{t<u\le t+\Delta} \frac{1}{\epsilon}\, E_t^\epsilon \int_u^T ds\, f_z(Z^\epsilon(u))\, E_u^\epsilon \int_0^1 d\tau \left[F_x(X^\epsilon(u^-) + \tau dY_d^\epsilon(u),\ \xi^\epsilon(s)) \cdot dY_d^\epsilon(u)\right].$$

Now, by a change of scale $s/\epsilon^2 \longrightarrow s$ and the use of (A3.1) and (A3.2) again, we see that this term is $O(\epsilon)\,E_t^\epsilon\,[Y_d^\epsilon(t+\Delta) - Y_d^\epsilon(t)]$.

Putting all the estimates together and cancelling the '$1/\epsilon$' term on the r.h.s. of (3.3) and the first term on the r.h.s. of (3.4), we get

$$(3.6)\qquad E_t^\epsilon f^\epsilon(t+\Delta) - f^\epsilon(t) = O(\Delta) + O(\epsilon)\,E_t^\epsilon\,(Y_c^\epsilon(t+\Delta) - Y_c^\epsilon(t)) + O(\epsilon)E_t^\epsilon\,(Y_d^\epsilon(t+\Delta) - Y_d^\epsilon(t)).$$

Eqn. (3.6) yields the quasimartingale and the uniformly (in ϵ) bounded in variation property on each interval $[0,T]$ for $f^\epsilon(\cdot)$. By letting $f(z) = z$ and noting that $Z_1^\epsilon(t) = O(\epsilon)$, we see that the theorem holds for the $\{Z^\epsilon(\cdot)\}$ component. Hence, it also holds for $\{X^\epsilon(\cdot)\}$, since $\sup_\epsilon E(Y_0^\epsilon(t) + Y_1^\epsilon(t)) < \infty$ for each t and

$$X^\epsilon(t) = (Z^\epsilon(t) + Z_1^\epsilon(t)) + Y^\epsilon(t) - Z_1^\epsilon(t). \qquad \text{Q. E. D.}$$

We summarize (3.3) to (3.6) for future use:

$$(3.7)\qquad E_t^\epsilon f^\epsilon(t+\Delta) - f^\epsilon(t) = \int_t^{t+\Delta} E_t^\epsilon f_z(Z^\epsilon(u))\, G(X^\epsilon(u), \xi^\epsilon(u))\, du$$

$$+ \frac{1}{\epsilon^2} E_t \int_t^{t+\Delta} du \int_u^T f_{zz}(Z^\epsilon(u))\, E_u^\epsilon F(X^\epsilon(u), \xi^\epsilon(s))\, ds\; F(X^\epsilon(u), \xi^\epsilon(u))$$

$$+ \frac{1}{\epsilon^2} E_t^\epsilon \int_t^{t+\Delta} du\, E_u^\epsilon \int_u^T f_z(Z^\epsilon(u))\, F_x(X^\epsilon(u), \xi^\epsilon(s))\, ds\; F(X^\epsilon(u), \xi^\epsilon(u)) + O(\Delta)$$

$$+ O(\epsilon) E_t^\epsilon (Y_c^\epsilon(t+\Delta) - Y_c^\epsilon(t)) + O(\epsilon) E_t^\epsilon (Y_d^\epsilon(t+\Delta) - Y_d^\epsilon(t)).$$

Theorem 3.1 implies that $\{X^\epsilon(\cdot) - Z_1^\epsilon,\ Y_0^\epsilon(\cdot),\ Y_1^\epsilon(\cdot),\ Z^\epsilon(\cdot) + Z_1^\epsilon(\cdot)\}$ are quasimartingales with uniformly bounded variation on each interval $[0,T]$ and are tight on $D^4[0,\infty]$ in the pseudo-path topology. Hence, the same tightness in the pseudo-path topology on $D^4[0,\infty]$ holds for $\{\bar{X}^\epsilon(\cdot)\}$. [To be consistent with the usage in [11] and in Section 2, we should set (w.l.o.g.) $X^\epsilon(t) = 0$ and $Z^\epsilon(t) = 0$ for sufficiently large t, but this is just a technicality which is convenient for the statements in [11] and does not affect the results.] In the next theorem, we choose and work with a weakly convergent subsequence, also indexed by ϵ and with the limit denoted by $\bar{X}(\cdot) = (X(\cdot), Y_0(\cdot), Y_1(\cdot), Z(\cdot))$. Clearly, the sample functions $Y_i(\cdot)$ can be taken to be non-decreasing elements of $D[0,\infty)$. Although $Y_i^\epsilon(0) = 0$, the limits $Y_i(\cdot)$ might not have value zero at $t = 0$. To account for this possible jump in the integrals, we use the normalization $Y_i(0^-) = 0$ in defining integrals with respect to the $Y_i(\cdot)$. (Recall that we are using $dY_i(s) = Y_i(s) - Y_i(s^-)$, since the $Y_i(\cdot)$ are taken to be right continuous.)

Since the pseudo-path topology is equivalent to convergence in measure, for almost all ω, t,

(3.8) $$X(t) = Z(t) + Y_0(t) - Y_1(t) .$$

In fact, (3.8) holds also at all t at which the functions are continuous. In the next theorem, we obtain the stronger and more useful result that there is a Wiener process $w(\cdot)$ such that $\bar{X}(\cdot)$ is non-anticipative with respect to $w(\cdot)$ and $(X(\cdot), Y_0(\cdot), Y_1(\cdot))$ satisfies (1.1) for that $w(\cdot)$. The limits $\bar{Y}_i(\cdot)$ would not be too useful were this not the case. We use the following 'ergodic' type assumptions.

A3.3. *There is a continuous function* $\bar{G}(\cdot)$ *such that for each* x ,

$$\frac{1}{N} \int_u^{u+N} E_u G(x , \xi(s))\, ds \xrightarrow{P} \bar{G}(x)$$

as u *and* N *go to* ∞ .

A3.4. *For* g *equal to either* F *or* F_x *and* $T > u + N$,

$$E \sup_x \left| \int_{u+N}^{T} E_u g(x , \xi(s))\, ds \right| \longrightarrow 0 ,$$

as u, N, T *go to* ∞ .

A3.5. *There is a continuous function* $\sigma(\cdot)$ *such that for each* x

$$\frac{1}{T_1} \int_u^{u+T_1} E_u\, F(x, \xi(\tau))\, d\tau \int_T^{T+T} F(x, \xi(s))\, ds$$

$$\xrightarrow{P} \sigma^2(x)/2$$

as T, u *and* T_1 *go to* ∞ . *Also, there is a continuous* $\bar{G}_0(\cdot)$ *such that for each* x

$$\frac{1}{T_1} \int_u^{u+T_1} E_u F(x , \xi(\tau))\, d\tau \int_{\tau}^{T+T} F_x(x , \xi(s))\, ds \xrightarrow{P} \bar{G}_0(x) .$$

Remark. If $\xi(\cdot)$ is stationary, then

$$\sigma^2(x) = \int_{-\infty}^{\infty} EF(x,\xi(0))\ F(x,\xi(s))ds,$$

$$\bar{G}_0(x) = \int_0^{\infty} EF(x,\xi(0))\ F_x(x,\xi(s))ds.$$

The requirements in (A3.4), (A3.5) are simply conditions on the rate of convergence as $\tau - u \to \infty$ of the conditional expectation of functions $g(\tau)$, of the noise data after time τ, given the data up to time u.

Theorem 3.2. *Assume* (A3.1) - (A3.5) *with* $\sup_\epsilon E_x[Y_0^\epsilon(t) + Y_1^\epsilon(t)] < \infty$ *for each* $t < \infty$. *Then* $\{Z^\epsilon(\cdot)\}$ *is tight in the Skorohod topology on* $D[0,\infty)$. *Any weak limit process is continuous* w.p.1. *Let* $\{X^\epsilon(\cdot),Y_0^\epsilon(\cdot),Y_1^\epsilon(\cdot),Z^\epsilon(\cdot)\}$ *be a weakly convergent subsequence in* $D^4[0,\infty)$., *with the pseudopath topology used on the first three components and the Skorohod topology on the last. Denote the limit by* $\bar{X}(\cdot) = (X(\cdot),Y_0(\cdot),Y_1(\cdot),Z(\cdot))$. *There is a standard Wiener process* $w(\cdot)$ *such that* $\bar{X}(\cdot)$ *is non-anticipative with respect to* $w(\cdot)$ and

$$Z(t) = \int_0^t \bar{G}\,(X(s))ds + \int_0^t \bar{G}_0(X(s))ds + \int_0^t \sigma(X(s))dw(s)\ . \tag{3.9}$$

Also, for all t, w.p.1,

$$X(t) = Z(t) + Y_0(t) - Y_1(t)\ . \tag{3.10}$$

[We need not require that $X^\epsilon(t) \in [0,\bar{B}]$ in this theorem.]

Proof. For purposes of the proof, the x-support of all functions can be taken to be compact, and the $Y_i^\epsilon(\cdot)$ uniformly bounded (i.e., are stopped on first reaching some large value N_0). The general case follows from this by taking appropriate limits on the bounds. Furthermore, we can now assume that $Z^\epsilon(\cdot)$ is bounded, and that the z-support of all functions is also compact.

Part (a). *Tightness of* $\{Z^\epsilon(\cdot)\}$ *in the Skorohod Topology.* Let $f(\cdot)$, $f_1^\epsilon(\cdot)$ and $f^\epsilon(\cdot)$ be defined as in Theorem 3.1. We use the perturbed test function method of [8] or [14] for proving tightness. Since $f_1^\epsilon(t) = O(\epsilon)$, Theorem 3.4 of [8] or Lemma 1 of [14] applied to the perturbed test function $f^\epsilon(\cdot)$ yields the tightness of $\{f(Z^\epsilon(\cdot))\}$ in $D[0,\infty)$ (Skorohod topology) for each smooth $f(\cdot)$, hence $\{Z^\epsilon(\cdot)\}$ is tight on $D[0,\infty)$ in the Skorohod topology.

(b) *The limit of* $\{Z^\epsilon(\cdot)\}$. Fix and work with a weakly convergent subsequence of $\{X^\epsilon(\cdot), Y_0^\epsilon(\cdot), Y_1^\epsilon(\cdot), Z^\epsilon(\cdot)\}$ also indexed by ϵ. The first three components converge in the pseudopath topology and the last in the Skorohod topology. Let $h(\cdot)$ be a bounded and continuous function. For $0 < k < \infty$ and $\Delta_i > 0$, $t_j \geq 0$, let us define $H(\cdot)$ by

$$H(\epsilon,\Delta_j,t_j,j \leq k) = h\left(\frac{1}{\Delta_j}\int_{t_j-\Delta_j}^{t_j} Y_i^\epsilon(s)ds,\ i = 0,1,\ \frac{1}{\Delta_j}\int_{t_j-\Delta_j}^{t_j} X^\epsilon(s)ds,\ \frac{1}{\Delta_j}\int_{t_j-\Delta_j}^{t_j} Z^\epsilon(s)ds,\ j \leq k\right).$$

Let $t_j \leq t < t + s$, for all j. We have (See (3.7))

$$\lim_\epsilon EH(\epsilon,\Delta_j,t_j,\ j \leq k)\left[f^\epsilon(t+s) - f^\epsilon(t) - \int_t^{t+s} E_t^\epsilon\,(T_1^\epsilon(u) + T_2^\epsilon(u) + T_3^\epsilon(u))du\right] = 0, \tag{3.11}$$

where

$$T_1^\epsilon(u) = f_z(Z^\epsilon(u))G(X^\epsilon(u),\ \xi^\epsilon(u)) \tag{3.12}$$

$$T_2^\epsilon(u) = \frac{1}{\epsilon^2}\int_u^T f_z(Z^\epsilon(u))\ E_u^\epsilon F_x(X^\epsilon(u),\ \xi^\epsilon(s))ds \cdot F(X^\epsilon(u),\ \xi^\epsilon(u))$$

$$T_3^\epsilon(u) = \frac{1}{\epsilon^2}\int_u^T f_{zz}(Z^\epsilon(u))\ E_u^\epsilon F(X^\epsilon(u),\ \xi^\epsilon(s))ds\ F(X^\epsilon(u),\ \xi^\epsilon(u))$$

Fix s. Let $\delta_\epsilon \to 0$ such that $\epsilon^2/\delta_\epsilon \to 0$ as $\epsilon \to 0$. Write $s = m_\epsilon\delta_\epsilon$ and suppose (w.l.o.g.) that the m_ϵ are integers. For the purpose of evaluating the limit in (3.11), we can drop the $f_1^\epsilon(\cdot)$ components of the $f^\epsilon(\cdot)$ (since they are $O(\epsilon)$). Also, for the same purpose, we can replace the integrals $\int_t^{t+s} E_t^\epsilon T_i^\epsilon(u)du$ by

$$E_t^\epsilon \sum_{j=0}^{m_\epsilon-1} \frac{1}{\delta_\epsilon} \cdot \delta_\epsilon \int_{t+j\delta_\epsilon}^{t+j\delta_\epsilon+\delta_\epsilon} E_{t+j\delta_\epsilon}^\epsilon\ T_i^\epsilon(u)du\ . \tag{3.13}$$

We work only with $T_3^\epsilon(\cdot)$, since the others are treated in essentially the same way. Rewrite (3.13) (with $i = 3$) as

$$(3.14)\quad E_t^\epsilon \sum_{j=0}^{m_\epsilon - 1} \delta_\epsilon \cdot \frac{1}{\delta_\epsilon \cdot \epsilon^2} \int_{t+j\delta_\epsilon}^{t+j\delta_\epsilon+\delta_\epsilon} E_{t+j\delta_\epsilon}^\epsilon f_{zz}(Z^\epsilon(u))du \int_0^T F(X^\epsilon(u),\xi^\epsilon(s))ds F(X^\epsilon(u),\xi^\epsilon(u)).$$

Now, change scale $u_{/\epsilon^2} \to u$, $s_{/\epsilon^2} \to s$ and define $s_j^\epsilon = (t+j\delta_\epsilon)/\epsilon^2$ and let (for notational convenience) $\hat{E}_j^\epsilon$ denote the expectation given $(Y_i^\epsilon(s),\ s \leq t + j\delta_\epsilon,\ i = 0,1,\ \xi(s),\ s \leq s_j^\epsilon)$. Then (3.14) can be rewritten as

$$(3.15)\quad E_t^\epsilon \sum_{j=0}^{m_\epsilon - 1} \delta_\epsilon \cdot \frac{\epsilon^2}{\delta_\epsilon} \int_{s_j^\epsilon}^{s_{j+1}^\epsilon} \hat{E}_j^\epsilon f_{zz}(Z^\epsilon(\epsilon^2 u))\ F(X^\epsilon(\epsilon^2 u),\ \xi(u))du \int_u^{T/\epsilon^2} F(X^\epsilon(\epsilon^2 u),\xi(s))ds \cdot$$

Due to (A3.4), the upper limit $T_{/\epsilon^2}$ can be replaced by any large T_1 and limits on T_1 taken after limits on ϵ are taken.

Fix δ-small. Let $\{B_i\}$ be disjoint intervals covering the range of $X^\epsilon(\cdot)$ and with diameter less than and let x_i denote an arbitrary point in B_i. Due to the upcrossing result for $\{X^\epsilon(\cdot) + Z_1^\epsilon(\cdot)\}$ implied by (2.2), and the fact that $Z_1^\epsilon(\cdot) = O(\epsilon)$, the fraction of the number of intervals in the set of intervals $\{[t+j\delta_\epsilon,\ t+j\delta_\epsilon+\delta_\epsilon),\ j \leq m_\epsilon\}$ for which $\sup_{u \leq \delta_\epsilon} |X^\epsilon(t+j\delta_\epsilon+u) - X^\epsilon(t+j\delta_\epsilon)| \geq \delta/2$ holds goes to zero in probability as $\epsilon \to 0$. Using this, the tightness in $D[0,\infty)$ (Skorohod topology) of $\{Z^\epsilon(\cdot)\}$ and (A3.4) yields that the limit of (3.16) as $\epsilon \to 0$ is the same as the limit of (3.15) as $\epsilon \to 0$.

$$(3.16)\quad E_t^\epsilon \sum_{j=0}^{m_\epsilon - 1} \delta_\epsilon \sum_i I_{\{X^\epsilon(t+j\delta_\epsilon) \in B_i\}} \cdot \frac{\epsilon^2}{\delta_\epsilon} \int_{s_j^\epsilon}^{s_{j+1}^\epsilon} du$$

$$f_{zz}(Z^\epsilon(t+j\delta_\epsilon))\hat{E}_j^\epsilon F(x_i,\xi(u)) \int_u^{T/\epsilon^2} F(x_i,\xi(s))ds$$

Now, (A3.5) implies that the limits are the same with (3.17) used in lieu of (3.16).

$$(3.17)\qquad E_t^{\epsilon} \sum_{j=0}^{m_\epsilon - 1} \delta_\epsilon \sum_i I_{\{X^\epsilon(t+j\delta_\epsilon)\in B_i\}} \frac{\sigma^2(x_i)}{2} f_{zz}(Z^\epsilon(t + j\delta_\epsilon))$$

Define the operator A(x) by

$$(3.18)\qquad A(x)f(z) = f_z(z)\bar{G}(x) + f_z(z)\,\bar{G}_0(x) + f_{zz}(z)\sigma^2(x)/2\ .$$

Finally, using the upcrossing result again to approximate the sum of the (indicator functions times δ_ϵ) by an integral, and putting the above estimate together, and using a similar method for T_1^ϵ and T_2^ϵ yields

$$(3.19)\qquad \lim_{\epsilon} EH(\epsilon,\Delta_j,t_j,\ j \leq k)\Big[f(Z^\epsilon(t+s)) - f(Z^\epsilon(t)) - \int_t^{t+s} A(X^\epsilon(u))f(Z^\epsilon(u))du\Big] = 0.$$

Since $Z^\epsilon(\cdot)$ *converges* in the Skorohod topology on $D[0,\infty)$ (and we have not yet proved the continuity of the limit $Z(\cdot)$) the set $\hat{I}$ of t-points for which $P\{Z(t) \neq Z(t^-)\} > 0$ is countable but need not be empty. Let t and t+s not be in $\hat{I}$. The triple $\{X^\epsilon(\cdot), Y_0^\epsilon(\cdot), Y_1^\epsilon(\cdot)\}$ converges in the pseudopath topology and the integrals in $H(\cdot)$ and in the brackets in (3.19) represent functions which are continuous with respect to convergence in measure. Then, taking limits in (3.19), we have

$$(3.20)\qquad EH(\Delta_j,t_j,j \leq k)\Big[f(Z(t+s)) - f(Z(t)) - \int_t^{t+s} A(X(u))f(Z(u))du\Big] = 0$$

where the function $H(\Delta_j,t_j,j \leq k)$ is defined to be just $H(\epsilon,\Delta_j,t_j,j \leq k)$ with all functions replaced by their limits as $\epsilon \to 0$.

Owing to the arbitrariness of k,t_j,Δ_j and $h(\cdot)$ and of the points s and t+s (not in $\hat{I}$), (3.20) implies that for each smooth $f(\cdot)$, the process defined by

$$f(Z(t)) - \int_0^t A(X(u))f(Z(u))du \equiv M_f(t)$$

is a martingale with respect to the sequence of σ-algebras generated by $\{X(s),Y_0(s),Y_1(s), Z(s), s \leq t\}$. The fact that the operator $A(x)$ is 'local' implies the continuity of $Z(\cdot)$. (See a proof of a related continuity result in [8], [14].)

If $f(z) = z$, the quadratic variation of $M_f(\cdot)$ is $\int_0^t \sigma^2(X(u))du$. Owing to these facts we can construct a standard Wiener process $w(\cdot)$ such that $X(\cdot),Z(\cdot),Y_0(\cdot)$ and $Y_1(\cdot)$ are non-anticipative with respect to $w(\cdot)$ and

$$(3.21) \qquad Z(t) = \int_0^t \left[\bar{G}(X(u)) + \bar{G}_0(X(u))\right]du + \int_0^t \sigma(X(u))dw(u) .$$

It follows from the continuity of $Z(\cdot)$ and the non-decreasing property of the $Y_i(\cdot)$ that we can define the limit $X(\cdot)$ of $\{X^\epsilon(\cdot)\}$ by (3.10).

Q.E.D.

4. The Discrete Parameter Problem.

The discrete parameter analog of Theorems 3.1 and 3.2 is obtained very similarly to the schemes used in those theorems, and we discuss only a few of the details, for one discrete parameter form. Just as for the continuous parameter case, the general ideas are applicable to a much broader class of processes than used here. Define $\{X_n^\epsilon\}$ by $X_0^\epsilon = x$ and

$$X_{n+1}^\epsilon = X_n^\epsilon + \epsilon G(X_n^\epsilon, \xi_n^\epsilon) + \sqrt{\epsilon} F(X_n^\epsilon, \xi_n^\epsilon) + \delta Y_n^\epsilon , \tag{4.1}$$

where we define $\delta Y_n^\epsilon = \delta Y_{0n}^\epsilon - \delta Y_{1n}^\epsilon$, and $\delta Y_{in}^\epsilon \geq 0$. Let E_n^ϵ denote the expectation conditioned on $\{X_j^\epsilon, j \leq n, \delta Y_{0j}^\epsilon, \delta Y_{1j}^\epsilon, \xi_j^\epsilon, j < n\}$. Define the processes $Y_i^\epsilon(\cdot)$ by $Y_i^\epsilon(t) = \sum_{j=0}^{n-1} \delta Y_{ij}$, $i = 0,1$, and $X^\epsilon(t) = X_n^\epsilon$ for $t \in [n\epsilon, n\epsilon+\epsilon)$. We will use

A4.1. $\sup_\epsilon E\,(Y_0^\epsilon(t) + Y_1^\epsilon(t)) < \infty$ *for each* t. $G(\cdot,\cdot)$, $F(\cdot,\cdot)$ *and* $F_x(\cdot,\cdot)$ *are bounded and measurable and the latter two functions are continuous in* x, *uniformly in* ξ.

A4.2. *For each* x, $EF(x,\xi_n^\epsilon) = 0$. *There is a* $K < \infty$ *such that for all* N *and* $n \leq N$,

$$\sup_{n,\epsilon,x} \left| \sum_{j=n}^{N} E_n^\epsilon\, g(x,\xi_j^\epsilon) \right| \leq K \tag{4.2}$$

where g *equals either* F or F_x.

Theorem 4.1. *Under* (A4.1) *and* (A4.2), $\{X^\epsilon(\cdot), Y_i^\epsilon(\cdot), i = 0,1\}$ *is (with the possible addition of a process* $X_1^\epsilon(\cdot) = O(\sqrt{\epsilon})$ *to* $X^\epsilon(\cdot)$*) a quasimartingale with variation uniformly bounded in* ϵ *on each interval* [0,T].

Proof. We proceed as in Theorem 3.1, and let $f(\cdot)$ be a function that is continuous and has bounded and continuous derivatives up to order three. Fix T, large. For $N = T_{/\epsilon}$, define

$$f_{1n}^{\epsilon} = \sqrt{\epsilon} \sum_{j=n}^{N} E_n^{\epsilon} f_z(Z_n^{\epsilon}) F(X_n^{\epsilon}, \xi_j^{\epsilon}),$$

where we define Z_n^{ϵ} by $Z_0^{\epsilon} = 0$ and $Z_{n+1}^{\epsilon} = Z_n^{\epsilon} + \epsilon G(X_n^{\epsilon}, \xi_n^{\epsilon}) + \sqrt{\epsilon}\, F(X_n^{\epsilon}, \xi_n^{\epsilon})$.

Define the processes $f_1^{\epsilon}(\cdot)$ and $Z_1^{\epsilon}(\cdot)$ by $f_1^{\epsilon}(t) = f_{1n}^{\epsilon}$ and $Z^{\epsilon}(t) = Z_n^{\epsilon}$ on the interval $[n\epsilon, n\epsilon + \epsilon)$. Define (as in Theorem 3.1) $f^{\epsilon}(t) = f(Z^{\epsilon}(t)) + f_1^{\epsilon}(t)$. We show that $f^{\epsilon}(\cdot)$ is a quasimartingale with the appropriately bounded variation. As in Theorem 3.1, we can suppose that $X^{\epsilon}(\cdot)$, $Z^{\epsilon}(\cdot)$ and $Y_i^{\epsilon}(\cdot)$ are bounded on [0,T].

With a rearrangement of terms, we can write

$$\begin{aligned}
(4.3) \qquad & E_n^{\epsilon} f^{\epsilon}(n\epsilon+\epsilon) - f^{\epsilon}(n\epsilon) = \\
& E_n^{\epsilon}[f(Z_n^{\epsilon} + \epsilon G(X_n^{\epsilon}, \xi_n^{\epsilon}) + \sqrt{\epsilon}\, F(X_n^{\epsilon}, \xi_n^{\epsilon})) - f(Z_n^{\epsilon})] \\
& \quad - \sqrt{\epsilon}\, f_z(Z_n^{\epsilon}) E_n^{\epsilon} F(X_n^{\epsilon}, \xi_n^{\epsilon}) \\
& + \sqrt{\epsilon} \sum_{j=n+1}^{N} E_n^{\epsilon}[f_z(Z_{n+1}^{\epsilon}) - f_z(Z_n^{\epsilon})] \cdot F(X_{n+1}^{\epsilon}, \xi_j^{\epsilon}) \\
& + \sqrt{\epsilon}\, f_z(Z_n^{\epsilon}) \sum_{j=n+1}^{N} E_n^{\epsilon}\,[F(X_{n+1}^{\epsilon}, \xi_j^{\epsilon}) - F(X_n^{\epsilon}, \xi_j^{\epsilon})]\,.
\end{aligned}$$

Via a truncated Taylor expansion, we see that the sum of the first two terms on the r.h.s. of (4.3) equal

$$(4.4) \qquad \epsilon\, f_z(Z_n^{\epsilon})\, E_n^{\epsilon}\, G(X_n^{\epsilon}, \xi_n^{\epsilon}) + \epsilon\, f_{zz}(Z_n^{\epsilon})\, E_n^{\epsilon}\, F^2(X_n^{\epsilon}, \xi_n^{\epsilon})/2 + O(\epsilon)..$$

Via a truncated Taylor expansion and (A4.2) the third term on the r.h.s. of (4.3) equals

$$(4.5) \qquad \epsilon\, f_{zz}(Z_n^\epsilon) \sum_{j=n+1}^{N} E_n^\epsilon F(X_{n+1}^\epsilon, \xi_j^\epsilon) F(X_n^\epsilon, \xi_n^\epsilon) + O(\epsilon) = O(\epsilon)$$

Similarly to what was done in Theorem 3.1 to the 'corresponding' integral, the last term in (4.3) can be written as

$$(4.6) \qquad \sqrt{\epsilon}\, f_z(Z_n^\epsilon) E_n^\epsilon \sum_{j=n+1}^{N} \int_0^1 E_{n+1}^\epsilon F_x(X_n^\epsilon + \tau(X_{n+1}^\epsilon - X_n^\epsilon), \xi_j^\epsilon) d\tau \cdot (X_{n+1}^\epsilon - X_n^\epsilon)$$

$$= O(\sqrt{\epsilon}) E_n^\epsilon (O(\sqrt{\epsilon}) + |\delta Y_n^\epsilon|) .$$

Putting all the estimates together yields.

$$E_n^\epsilon f^\epsilon(n\epsilon+\epsilon) - f^\epsilon(n\epsilon) = O(\epsilon) + O(\sqrt{\epsilon}) E_n^\epsilon |\delta Y_n^\epsilon| .$$

Letting $f(z) = z$ yields the desired result, since $f_1^\epsilon(\cdot) = O(\sqrt{\epsilon})$ and $\sup_\epsilon E(Y_0^\epsilon(T) + Y_1^\epsilon(T)) < \infty$.

Q.E.D.

Theorem 3.2 can also be carried over to the discrete parameter case. We will use the conditions.

A4.3. $G(\cdot,\xi)$ *is continuous in* x, *uniformly in* ξ. There is a continuous $\bar{G}(\cdot)$ *such that for each* x

$$\frac{1}{N} \sum_{n}^{n+N} E_n^\epsilon G(x, \xi_j^\epsilon) \xrightarrow{P} \bar{G}(x)$$

as n *and* N *go to* ∞ .

A4.4. *There are continuous* $R(j,x)$ *and* $R_0(j,x)$ *such that for each* x

$$\frac{1}{N}\sum_{n=m}^{m+N} E_m^{\epsilon} F(x,\xi_{n+j}^{\epsilon})F(x,\xi_n^{\epsilon}) \xrightarrow{P} R(j,x)$$

$$\frac{1}{N}\sum_{n=m}^{m+N} E_m^{\epsilon} F_x(x,\xi_{n+j}^{\epsilon})F(x,\xi_n^{\epsilon}) \xrightarrow{P} R_0(j,x) ,$$

as m, N *and* $n-m$ *go to* ∞ .

A4.5. *For* g *equal to either* F *or* F_x,

$$E \sup_x \left| \sum_{n+N_1}^{N} E_n^{\epsilon} g(x,\xi_j^{\epsilon}) \right| \to 0$$

as N,n *and* N_1 go to ∞ *(with* $N > n+N_1$) .

Define

$$\sigma^2(x) = R(0,x) + 2\sum_{1}^{\infty} R(j,x) = \sum_{-\infty}^{\infty} R(j,x)$$

$$\bar{G}_0(x) = \sum_{1}^{\infty} R_0(j,x) .$$

It can be shown that (A4.5) implies that the sums $\sum_1^n R(j,x)$, $\sum_1^n R_0(j,x)$ converge uniformly in x as $n \to \infty$ (not necessarily absolutely).

A proof parallel to that of Theorem 3.2 yields

Theorem 4.2. *Assume* (A4.1) *to* (A4.5). *Then the conclusions of Theorem* 3.2 *hold for the model* (4.1).

5. Auxiliary Results

In this section, we obtain some estimates which will be useful in Section 6, for the proofs of the convergence of the costs $V_0^\epsilon(x,Y_0^\epsilon,Y_1^\epsilon)$ to either cost $V_0(x,Y_0,Y_1)$ or $V(x)$. We will show, for several reasonable classes of control policies, that $\sup_\epsilon E|Y_i^\epsilon(t)|^k < \infty$ for each $k > 0$ and $t < \infty$. This implies the uniform integrability property needed in the next section: In Section 6, we will need to know that the sequence of optimal or δ-optimal controls for $x(\cdot)$ are uniformly integrable. similarly, we will need to know whether the sequence of optimal or δ-optimal controls for $x^\epsilon(\cdot)$ is uniformly integrable.

The symbol τ_ϵ will denote a stopping time with respect to either of the 'data' σ-algebras $B\{\xi^\epsilon(s), s \leq t\} \equiv B_t^\epsilon$ or $B\{\xi_n^\epsilon, \epsilon n < t\} \equiv B_t^\epsilon$ depending on the case, and we write $E_{\tau_\epsilon}^\epsilon$ and $P_{\tau_\epsilon}^\epsilon$ for the expectation and probability, conditioned on the data up to time τ_ϵ.

Theorem 5.1. *Assume either* (A3.1), (A3.2) *or* (A4.1), (A4.2). *Let* $Q_\epsilon(\cdot)$ *and* $Q_{\epsilon n}$ *be bounded and* B_t^ϵ *measurable (for* $n\epsilon < t$, *in the latter case). Define* $X^\epsilon(\cdot)$ *and* X_n^ϵ by

$$dX^\epsilon = [G(X^\epsilon,\xi^\epsilon) + F(X^\epsilon,\xi^\epsilon)/\epsilon + Q_\epsilon/\epsilon]dt \tag{5.1a}$$

$$X_{n+1}^\epsilon = X_n^\epsilon + \epsilon G(X_n^\epsilon,\xi_n^\epsilon) + \sqrt{\epsilon}\, F(X_n^\epsilon,\xi_n^\epsilon) + \sqrt{\epsilon}\, Q_{\epsilon n}\,. \tag{5.1b}$$

Define $Z^\epsilon(\cdot)$ *as in Section* 3 *or* 4 *(continuous and discrete parameters case, resp.). For integer* k *and* $t < \infty$, *there are* $\infty > K_{kt} \to 0$ *as* $t \to 0$ *such that (for small* $\epsilon > 0$)

$$E \sup_{s \leq t} |Z^\epsilon(\tau_\epsilon + s) - Z^\epsilon(\tau_\epsilon)|^{2k} \leq K_{2kt} \tag{5.2}$$

for all finite (w.p.1) τ_ϵ.

Proof. For arbitrary $T < \infty$, define

$$f_1^{2k\epsilon}(t) = \int_t^T 2kZ^\epsilon(t)^{2k-1}E_t^\epsilon\, F(X^\epsilon(t),\xi^\epsilon(s))ds/\epsilon = O(\epsilon)\left|Z^\epsilon(t)\right|^{2k-1}.$$

The right hand equality is a consequence of (A3.2) and the change of variable $s/\epsilon^2 \to s$. We do the proof only for the continuous parameter case and for $\tau_\epsilon = G = Z^\epsilon(0) = 0$ for simplicity. The proof of the other cases is essentially the same.

Analogous to what was done to get (3.4) and (3.7), we have

$$E_t^\epsilon\left[Z^\epsilon(t+s)^{2k} + f_1^{2k\epsilon}(t+s)\right] - \left[Z^\epsilon(t)^{2k} + f_1^{2k\epsilon}(t)\right] = E_t^\epsilon \int_t^{t+s} c_0^{2k\epsilon}(u)du, \tag{5.3}$$

where

$$\begin{aligned} c_0^{2k\epsilon}(u) &= \frac{1}{\epsilon^2}\int_u^T 2k[(2k-1)Z^\epsilon(u)^{2k-2}E_u^\epsilon F(X^\epsilon(u),\xi^\epsilon(s))ds]F(x^\epsilon(u),\xi^\epsilon(u)) \\ &\quad + \frac{1}{\epsilon^2}\int_u^T 2kZ^\epsilon(u)^{2k-1}E_u^\epsilon F_x(x^\epsilon(u),\xi^\epsilon(s))ds[Q^\epsilon(u) + F(X^\epsilon(u),\xi^\epsilon(u))] \\ &= O(1)\int_{u/\epsilon^2}^{T/\epsilon^2} Z^\epsilon(u)^{2k-2}E_u^\epsilon F(X^\epsilon(u),\xi(s))ds \\ &\quad + O(1)\int_{u/\epsilon^2}^{T/\epsilon^2} Z^\epsilon(u)^{2k-1}E_u^\epsilon F_x(x^\epsilon(u),\xi(s))ds. \end{aligned}$$

By (A3.2), we can write this expression as

$$\begin{aligned} &Z^\epsilon(u)^{2k-2}\, C_2^{2k\epsilon}(u) + Z^\epsilon(u)^{2k-1}\, C_1^{2k\epsilon}(u) \\ &\qquad = O(1)\,(\left|Z^\epsilon(u)\right|^{2k-2} + \left|Z^\epsilon(u)\right|^{2k-1}), \end{aligned} \tag{5.4}$$

where the $C_i^{k\epsilon}(\cdot)$ are defined in the obvious way and are bounded.

By (5.3), (5.4) and the bound $f_1^{2k\epsilon}(\cdot)$, there are $\infty > K_{2k}(\tau) \to 0$ as $\tau \to 0$ and constants K'_{2k} such that, for $t \le \tau$,

$$E\left[Z^\epsilon(t)^{2k} + f_1^{2k\epsilon}(t)\right] \le K'_{2k}\int_0^t E\left[1 + \left|Z^\epsilon(s)\right|^{2k-1}\right]ds \le K_{2k}(\tau).$$

Define

$$M^{2k\epsilon}(t) = \left[Z^\epsilon(t)^{2k} + f_1^{2k\epsilon}(t)\right] - \int_0^t \left[Z^\epsilon(s)^{2k-2}C_2^{2k\epsilon}(s) + Z^\epsilon(s)^{2k-1}C_1^{2k\epsilon}(s)\right]ds.$$

By (5.3), $M^{2k\epsilon}(\cdot)$ is a martingale. From the above estimates there are functions $K^n_{2k}(t) \to 0$ as $t \to 0$ such that (use Doobs inequality [15, Theorem 7.3.4]

$$(5.5) \qquad E \sup_{s \le t} |M^{2k\epsilon}(s)|^2 \le 4E|M^{2k\epsilon}(t)|^2 \le K^n_{2k}(t).$$

By the bound on $f_1^{2k\epsilon}(\cdot)$ and (5.5), we get (5.2) for $\tau_\epsilon = 0$. Q.E.D.

Theorem 5.2. *Assume the conditions of Theorem* 5.1, *except with* $Q_\epsilon(t) = 0$ (*or* $Q_{\epsilon n} = 0$) *for* $t \ge \tau_\epsilon$. *Given* $\Delta_0 > 0$, *there are* $\delta_0 > 0$ *and* $T_0 > 0$ *such that for all small* ϵ

$$(5.6) \qquad P^\epsilon_{\tau_\epsilon} \left\{ \sup_{t \le T_0} |Z^\epsilon(\tau_\epsilon + t) - Z^\epsilon(\tau_\epsilon)| \ge \Delta_0 \right\} \le 1 - \delta_0 .$$

Proof. The result follows from Theorem 5.1.

Recall the definition of $\bar{B}$ in Section 1. We now describe some classes of controls and obtain some estimates of path excursions under the controls. Let L and U be numbers such that $0 \le L < U \le \bar{B}$. Define

$$(5.7) \qquad \begin{aligned} dY^\epsilon_0(t) &= \left[F(X^\epsilon(t),\xi^\epsilon(t))/\epsilon + G(X^\epsilon(t),\xi^\epsilon(t))\right]^- dt\, I_{\{X^\epsilon(t)=L\}} \\ dY^\epsilon_1(t) &= \left[F(X^\epsilon(t),\xi^\epsilon(t))/\epsilon + G(X^\epsilon(t),\xi^\epsilon(t))\right]^+ dt\, I_{\{X^\epsilon(t)=U\}}. \end{aligned}$$

For obvious reasons, we call this the (L,U) *barrier control* (following the usage in [4]). Define the discrete parameter barrier policy in the analogous way: the $dY^\epsilon_i(\cdot)$ and $\delta Y^\epsilon_{i\,n}$ are just large enough to keep $X^\epsilon(\cdot)$ and X^ϵ_n in the set [L,U]. The dY^ϵ_i/dt will be one of the candidates for the Q_ϵ in Theorem 5.1.

Let $\Delta_0 < \bar{B}/2$. We define a specific control policy - called the $(\bar{B},\Delta_0)$-control (for the continuous parameter case) as follows. If $X^\epsilon(t^-) = \bar{B}$, immediately set $Y^\epsilon_1(t) = Y^\epsilon_1(t^-) + \Delta_0$ and $X^\epsilon(t) = \bar{B} - \Delta_0$. Also, $Y^\epsilon_0(\cdot)$ increases just fast enough to keep $X^\epsilon(t) \ge 0$; i.e., $Y^\epsilon_0(\cdot)$ is given by (5.7) for L = 0. There are analogous definitions

and results for the discrete parameter process. The $(\bar{B},\Delta_0)$-control has some nice properties which render it useful for the discussion in the next section.

Theorem 5.3. *Assume the* $(\Delta_0,\bar{B})$*-control and either* (A3.1), (A3.2) *or* (A4.1), (A4.2). *For each* t *and integer* k,

$$\sup_{\epsilon,\tau_\epsilon} E\left|Y_i^\epsilon(\tau_\epsilon+t) - Y_i^\epsilon(\tau_\epsilon)\right|^k < \infty . \tag{5.8}$$

Remark. Owing to the conditioning in (5.6), the estimates for Y_i^ϵ are proved almost as if the 'return' process from the point $(\bar{B}-\Delta_0)$ to (either $\bar{B}$ or $\bar{B}-2\Delta_0$), then back to $\bar{B}-\Delta_0$, etc., were constructed from a Bernoulli sequence.

Proof. We do the continuous parameter case only, and i = 1. The case i = 0 is treated by an argument based on Theorems 5.2 and 5.4. W.l.o.g., set $\tau_\epsilon = 0$.

Define the stopping times: $\sigma_0^\epsilon = \min\{t \geq 0 : X^\epsilon(t) = \bar{B} - \Delta_0\}$ and for $i > 0$, $\rho_i^\epsilon = \min\{t > \sigma_{i-1}^\epsilon : |X^\epsilon(t) - (\bar{B} - \Delta_0)| \geq \Delta_0\}$, $\sigma_i^\epsilon = \min\{t \geq \rho_i^\epsilon : X^\epsilon(t) = \bar{B} - \Delta_0\}$. We will estimate the k^{th} moment of $N^\epsilon(t) = \max\{i : \sigma_i^\epsilon \leq t\}$. Define the $\{0,1\}$ valued random variable U_i^ϵ as follows. Use the T_0 of Theorem 5.2. If $\rho_i^\epsilon - \sigma_{i-1}^\epsilon < T_0$, set $U_i^\epsilon = 0$ and call the event a 'failure'. If $\rho_i^\epsilon - \sigma_{i-1}^\epsilon \geq T_0$, set $U_i^\epsilon = 1$ and call the event a 'success'. Let N_i^ϵ denote the number of successive passages of $X^\epsilon(\cdot)$ from $\bar{B}-\Delta_0$ to either $\bar{B}$ or $\bar{B} - 2\Delta_0$ which are failures, after the i^{th} success. Then

$$N^\epsilon(t) \leq \frac{t}{T_0} + \sum_0^{t/T_0-1} N_i^\epsilon .$$

There are $K_k < \infty$ such that

$$N^\epsilon(t)^k \leq K_k(t/T_0)^k + K_k(t/T_0)^k \sum_0^{t/T_0-1} (N_i^\epsilon)^k .$$

We will bound $E(N_i^\epsilon)^k$. Let $\sigma_{s_i}^\epsilon$ denote the return time of $X^\epsilon(\cdot)$ to $\bar{B} - \Delta_0$ immediately after the i^{th} success. Then, by Theorem 5.2,

$$P^\epsilon_{\sigma_{s_i}^\epsilon}\{N_i^\epsilon \geq n\} =$$

$$P^\epsilon_{\sigma_{s_i}^\epsilon}\{\sigma^\epsilon_{s_i+j+1} - \sigma^\epsilon_{s_i+j} < T_0,\ j < n\}$$

$$\leq (1 - \delta_0)^{n-1}$$

This yields $E(N_i^\epsilon)^k \leq (\text{constant})/\delta_0$, and the proof is concluded, since $Y_1^\epsilon(t) \leq \Delta_0 N^\epsilon(t)$.

Q.E.D

Theorem 5.4. *Assume either* (A3.1), (A3.2) *or* (A4.1), (A4.2) *and the* (L,U)*-barrier policy. Then for each* t

$$\sup_{\epsilon,T}\left[E(Y_0^\epsilon(t+T) - Y_0^\epsilon(T))^2 + E(Y_1^\epsilon(t+T) - Y_1^\epsilon(T))^2\right] < \infty . \tag{5.9}$$

Proof. Again, we do only some of the details for $Y_0^\epsilon(\cdot)$, and for the continuous parameter case. Drop the $G(\cdot,\cdot)$ for notational simplicity. Denote the initial time by t_0 and let $\Delta_0 < (U-L)/2$ and define the stopping times

$$\sigma_0^\epsilon = \min\{t \geq t_0 : X^\epsilon(t) = L\}$$

and for $i > 0$,

$$\sigma_i^\epsilon = \min\{t > \rho_i^\epsilon : X^\epsilon(t) = L\},\ \rho_i^\epsilon = \min\{t > \sigma_{i-1}^\epsilon : X^\epsilon(t) = L + \Delta_0\} .$$

Set the stopping times to ∞ if they are not otherwise defined. All the needed estimates can be shown to be uniform in t_0 and we set $t_0 = 0$ for simplicity.

We can write (and simultaneously define $\hat{Z}^\epsilon(\cdot)$)

$$(5.10)\qquad \sum_i \left[X^\epsilon(\rho_{i+1}^\epsilon \cap t) - X^\epsilon(\sigma_i^\epsilon \cap t)\right] =$$

$$\sum_i \left[Z^\epsilon(\rho_{i+1}^\epsilon \cap t) - Z^\epsilon(\sigma_i^\epsilon \cap t)\right] + Y_0^\epsilon(t)$$

$$= \hat{Z}^\epsilon(t) + Y_0^\epsilon(t) .$$

The mean square value of the term on the left of (5.10) is bounded above by Δ_0^2 times the expectation of the square of the number of i for which $\rho_i^\epsilon \leq t$. By an argument very similar to that used in Theorem 5.3, this can be shown to be bounded uniformly in ϵ for each t.

Define $M^\epsilon(t) = Z_1^\epsilon(t) - \int_0^t C_1^\epsilon(s)ds$, where $Z_1^\epsilon(\cdot)$ is defined in Theorem 3.1 or, equivalently, it is the $f_1^{1\epsilon}(\cdot)$ of Theorem 5.1. The $C_1^\epsilon(\cdot)$ is defined in Theorem 5.1. The $C_2^{1\epsilon}(\cdot)$ defined there doesn't appear here, since $2k = 1$ here. Define $N^\epsilon(\cdot)$ as in Theorem 5.3. Then, since $M^\epsilon(\cdot)$ is a martingale on the interval where $dY_1^\epsilon(\cdot) = 0$ we have,

$$E\Big(\sum_i \left[M^\epsilon(\rho_{i+1}^\epsilon \cap t) - M^\epsilon(\sigma_i^\epsilon \cap t)\right]\Big)^2$$

$$= \sum_i E\left|M^\epsilon(\rho_{i+1}^\epsilon \cap t) - M^\epsilon(\sigma_i^\epsilon \cap t)\right|^2$$

$$(5.11)\qquad = O(1)\ E\Big(\sup_{s\leq t} |Z^\epsilon(s)|^2 + 1\Big)\ N^\epsilon(t)$$

$$= O(1) E^{1/2}\Big(\sup_{s\leq t} |Z^\epsilon(s)|^4 + 1\Big)\ E^{1/2}\,|N^\epsilon(t)|^2 \leq K_1 < \infty .$$

The last inequality follows from Theorems 5.1 and 5.3. Since $C_1^{1\epsilon}(\cdot) = O(1)$, and $Z_1^\epsilon(\cdot) = O(\epsilon)$, and $\sup_\epsilon E(N^\epsilon(t))^2 < \infty$, there are $K_2 < \infty$, $K_3 < \infty$, such that the left side of (5.11) can be bounded *below* by

$$K_2\, E\Big(\sum_i \left[Z^\epsilon(\rho_{i+1}^\epsilon \cap t) - Z^\epsilon(\sigma_i^\epsilon \cap t)\right]\Big)^2 - K_3$$

$$= K_2\, E\left|\hat{Z}^\epsilon(t)\right|^2 - K_3 .$$

The proof that $\sup E\left|\hat{Z}^\epsilon(t)\right|^2 < \infty$ follows from these inequalities. Q.E.D

6. Convergence of the Costs and Controls.

In [1], it is shown that there are $0 \leq L^* < U^* < \infty$ such that (under appropriate conditions) the optimal control for (1.1) is a (L^*,U^*)-barrier control. We assume that $\bar{B}$ is large enough so that $U^* \leq \bar{B}$. Let $\bar{Y}_i(\cdot)$, $i = 0,1$, denote this optimal control. The set of increments of the 'local time' control processes $\{\bar{Y}_i(n+1) - \bar{Y}_i(n), i = 0,1, n < \infty\}$ are uniformly integrable. Let $\bar{Y}_i^\epsilon(\cdot)$, $i = 0,1$, denote the (L^*,U^*)-barrier control for $X^\epsilon(\cdot)$ (continuous or discrete time). The following theorem says that the optimal control for $X(\cdot)$ is 'nearly' optimal for $X^\epsilon(\cdot)$.

Theorem 6.1. *Assume either* (A3.1) *to* (A3.5) *or* (A4.1) *to* (A4.5). *Let* (1.1) *have a unique weak sense solution for the* (L^*,U^*)*-barrier policy, and let this policy be optimal. Then* $\{X^\epsilon(\cdot), \bar{Y}_0^\epsilon(\cdot), \bar{Y}_1^\epsilon(\cdot)\} \Rightarrow (X(\cdot), \bar{Y}_0(\cdot), \bar{Y}_1(\cdot))$ *in the pseudopath topology, and there is a Wiener process* $w(\cdot)$ *such that* $(X(\cdot), \bar{Y}_0(\cdot), \bar{Y}_1(\cdot))$ *is non-anticipative with respect to* $w(\cdot)$, *and* (1.1) *holds. Also*

$$V_0^\epsilon(x, \bar{Y}_0^\epsilon, \bar{Y}_1^\epsilon) \to V_0(x, \bar{Y}_0, \bar{Y}_1) = V(x). \tag{6.1}$$

For $\delta > 0$, *let* $\hat{Y}_0^\epsilon(\cdot)$, $\hat{Y}_1^\epsilon(\cdot)$ *be* δ*-optimal policies for* $X^\epsilon(\cdot)$ *such that* (6.2) *is uniformly integrable.*

$$\{\hat{Y}_i^\epsilon(n+1) - \hat{Y}_i^\epsilon(n), \epsilon > 0, n < \infty\} \tag{6.2}$$

Then

$$\delta + \lim_\epsilon V^\epsilon(x) \geq \lim_\epsilon V_0^\epsilon(x, \hat{Y}_0^\epsilon, \hat{Y}_1^\epsilon) \geq V(x) . \tag{6.3}$$

Remark on (6.2). The uniform integrability is used basically to assure that the cost associated with the limit process is the limit of the costs associated with $X^\epsilon(\cdot)$. We have not been able to prove the theorem without it, unless all cost terms are positive (see Theorem 6.2). With the cost structure used here and in [1], it is conceivable (in that we have not yet proved otherwise) that as $\epsilon \to 0$, the increments in both $\hat{Y}_0^\epsilon(\cdot)$ and $\hat{Y}_1^\epsilon(\cdot)$ grow without bound. But, (as shown in Section 5) this won't happen for a large class of reasonable controls. The uniform integrability holds for a wide variety of control processes: E.g., for (1) Combinations of the (L,U)-barrier and $(\bar{B},\Delta_0)$-policies (Theorems 5.3 and 5.4); (2) these theorems can be extended to cover the case where there are numbers L^0, U^0, Δ_0, Δ_1 where $\Delta_0 + \Delta_1 < (U^0 - L^0)/2$ and $Y_0^\epsilon(\cdot)$ acts only in $[L^0, L^0 + \Delta_0]$, $Y_1^\epsilon(\cdot)$ only in $[U^0 - \Delta_0, U^0]$, and with maximum jump $\leq \Delta_1$; (3) Let $Y_i^\epsilon(\cdot)$ denote any admissible policy and fix N. Define

$$\tau_n^\epsilon = \min\{t > n: (Y_0^\epsilon(t) - Y_0^\epsilon(n)) + (Y_1^\epsilon(t) - Y_1^\epsilon(n)) \geq N\} \cap (n + 1).$$

On the interval [n, n+1), use $Y_i^\epsilon(\cdot)$ on $[n, \tau_n^\epsilon)$, then switch to a barrier or $(\bar{B},\Delta_0)$ policy on $[\tau_n^\epsilon, n+1)$. In Theorem 6.2, it is shown that (6.2) is not needed if $-k_1 dY_1(t)$ is replaced by the positive cost increment $k_1 dY_1(t)$.

Proof. We do only the continuous parameter case. Let $X^\epsilon(\cdot)$ denote the process with the $\bar{Y}^\epsilon(\cdot)$ used. By Theorem 5.4, $\{\bar{Y}_i^\epsilon(n+1) - \bar{Y}_i^\epsilon(n), \epsilon > 0, n < \infty\}$ is uniformly integrable. Extract a weakly convergent subsequence of $\{X^\epsilon(\cdot), \bar{Y}_0^\epsilon(\cdot), \bar{Y}_1^\epsilon(\cdot)\}$ (pseudopath topology) and denote the limit by $(X(\cdot),Y_0(\cdot),Y_1(\cdot))$. By Theorem 3.2, this triple satisfies (1.1) for some $w(\cdot)$. Clearly, the $\bar{Y}_0(\cdot),\bar{Y}_1(\cdot)$ is the (L^*,U^*)-barrier policy, since it can only increase when $X(t) = L^*$ or U^* as appropriate. Hence $Y_i(\cdot) = \bar{Y}_i(\cdot)$. By this and the uniqueness of the solution to (1.1), the limit does not depend on the chosen subsequence.

By the uniform integrability asserted in the above paragraph,

$$\lim_{\epsilon} V_0^{\epsilon}(x,\bar{Y}_0^{\epsilon},\bar{Y}_1^{\epsilon}) = \lim_{\epsilon} E \int_0^{\infty} e^{-\beta t}\left[k_0 d\bar{Y}_0^{\epsilon}(t) - k_1 d\bar{Y}_1^{\epsilon}(t) + k(X^{\epsilon}(t))dt\right] \tag{6.4}$$

$$\rightarrow E \int_0^{\infty} e^{-\beta t}\left[k_0 d\bar{Y}_0(t) - k_1 d\bar{Y}_1(t) + k(X(t))dt\right]$$

$$= V_0(x,\bar{Y}_0,\bar{Y}_1) = V(x) .$$

To get (6.3), repeat the procedure with controls $Y_0^{\epsilon}(\cdot)$, $Y_1^{\epsilon}(\cdot)$. Here, the limit $(X(\cdot),Y_0(\cdot),Y_1(\cdot))$ might depend on the chosen subsequence. But, for any convergent subsequence $\{\epsilon_n\}$ we get $\lim_{\epsilon=\epsilon_n\to 0} V_0^{\epsilon}(x,Y_0^{\epsilon},Y_1^{\epsilon}) \rightarrow V_0(x,Y_0,Y_1) \geq V(x)$. Hence, by the definition of δ-optimality and the weak convergence,

$$\delta + \lim_{\epsilon} V^{\epsilon}(x) \geq \lim_{\epsilon} V_0^{\epsilon}(x,\hat{Y}_0^{\epsilon},\hat{Y}_1^{\epsilon})$$

$$\geq \inf_{Y_0,Y_1} V_0(x,Y_0,Y_1) = V(x).$$

Q.E.D.

Theorem 6.2. *Assume the conditions of Theorem* 6.1, *except for the uniform integrability of* (6.2), *but let the cost be*

$$E_x \int_0^{\infty} e^{-\beta t}[k_0 dY_0(t) + k_1\, dY_1(t) + k(x(t))dt] = V_0(x,Y_0,Y_1),$$

and similarly define $V_0^{\epsilon}(x,Y_0^{\epsilon},Y_1^{\epsilon})$, *where* $k_i > 0$. *Then the conclusions of Theorem* 6.1 (*with the* δ *in* (6.3) *replaced by* 2δ) *hold.*

Proof. Let $\hat{Y}_i^{\epsilon}(\cdot)$, $i = 0,1$, denote a δ-optimal policy. We can suppose that $\sup_{\epsilon}[E_x Y_0^{\epsilon}(t) + E_x Y_1^{\epsilon}(t)] < \infty$ for each $t < \infty$ and

$$\lim_{T} \sup_{\epsilon} \int_T^{\infty} e^{-\beta t}[k_0 dY_0^{\epsilon}(t) + k_1 dY_1^{\epsilon}(t) + k(X^{\epsilon}(t))dt] = 0,$$

since this holds for any barrier policy. In fact, there is a $N_\delta < \infty$ such that if we switch to the (L^*,U^*) barrier policy (or to any barrier policy) once the $Y_i^\epsilon(t)$ exceeds N_δ, we change the cost by less than δ. But, then the set (6.2) is uniformly integrable, and Theorem 6.1 holds.

Q.E.D.

7. Average Cost Per Unit Time.

The methods of Sections 1 to 5 can be used to adjust the proof of Theorem 8 in [6] to get the result which is analogous to Theorem 6.1 for the average cost per unit time problem. Only an outline of the method will be given. The reader is referred to the reference for more details on the structure of the approximation for the average cost problem for the non-singular case (and which can be carried over to our case).

For the average cost per unit time problem, we wish to work with feedback controls and, hence, use only $Y_i^\epsilon(\cdot)$, $i = 0,1$, or $Y_i(\cdot)$, $i = 0,1$, for which the associated processes $\xi^\epsilon(\cdot)$ and $(X^\epsilon(\cdot),\xi^\epsilon(\cdot))$ or $X(\cdot)$, resp., are bounded Markov-Feller processes. Also, let $\{\xi^\epsilon(t),\ \epsilon > 0,\ t < \infty\}$ be bounded. The cost criteria are

$$\overline{\lim_T}\ E\ \tfrac{1}{T}\int_0^T \left[k_0 dY_0(t) - k_1 dY_1(t) + k(X(t))dt\right] \equiv \gamma(Y_0,Y_1)$$

$$\overline{\lim_T}\ E\ \tfrac{1}{T}\int_0^T \left[k_0 dY_0^\epsilon(t) - k_1 dY_1(t) + k(X^\epsilon(t))dt\right] = \gamma^\epsilon(Y_0,Y_1)\ .$$

For simplicity, we do only the continuous parameter case. The discrete parameter case uses very similar assumptions and proof. Let PM (PM^ϵ,resp.) denote the class of feedback control processes for which $X(\cdot)$ (resp., $(X^\epsilon(\cdot),\xi^\epsilon(\cdot))$ is a Markov-Feller process. Let NA (resp., NA^ϵ) denote the class of non-anticipative controls. We will use the following assumptions.

A7.1. *There is an* $\epsilon_0 > 0$ *such that for each* $\delta > 0$ *and* $\epsilon \leq \epsilon_0$, *there are* δ*-optimal controls* $\in PM^\epsilon$ *of the form*

(7.1) $\quad dY_i^\epsilon = Q_i^\epsilon(x,\xi)dt, \quad i = 0,1,$

where the $Q_i^\epsilon(\cdot,\cdot)$ *are continuous.*

Note: If $Q_i^\epsilon(x,\xi)$ *is Lipschitz continuous in* x, *uniformly in* ξ, *then* $Y_i^\epsilon(\cdot) \in PM^\epsilon$. *See the remark below where it is shown that the barrier and* $(\bar{B},\Delta_0)$ *policies can often be smoothed to yield a continuous* $Q_i^\epsilon(\cdot,\cdot)$.

A7.2. *A* (L^*,U^*) *barrier control* $\bar{Y}_i(\cdot)$ *is optimal for* (1.1), *and* (1.1) *then has a unique invariant measure. This control is in* PM *and its adaptation* $\bar{Y}_i^\epsilon(\cdot)$, i = 0,1, *to* $X^\epsilon(\cdot)$ *is in* PM^ϵ. *When applied to* PM^ϵ, $(X^\epsilon(\cdot),\xi^\epsilon(\cdot))$ *has a unique solution and invariant measure.*

A7.3. $\quad \inf_{Y_i \in PM} \gamma(Y_0,Y_1) = \inf_{Y_i \in NA} \gamma(Y_0,Y_1)$.

Theorem 7.1. *Assume* (A3.1) *to* (A3.5) *and* (A7.1) *to* (A7.3). *Let* $G(x,\xi)$ *be Lipschitz continuous in* x, *uniformly in* ξ. *For* $\delta > 0$, *let* $Y_i^\epsilon(\cdot)$, i = 1,2, *be a sequence of* δ*-optimal controls in* PM^ϵ *(for* $X^\epsilon(\cdot)$*) and let (the* $Y_i^\epsilon(\cdot)$ *are of the form discussed in* (A7.1)*) with* Q_i^ϵ *associated with* Y_i^ϵ

(7.2) $\quad \{Y_i^\epsilon(n+1) - Y_i^\epsilon(n),\ \epsilon > 0,\ n < \infty,\ X^\epsilon(0) = x,\ \xi^\epsilon(0) = \xi\}$

be uniformly integrable. Then

$$\delta + \lim_\epsilon \gamma^\epsilon(Y_0^\epsilon,Y_1^\epsilon) \geq \lim_\epsilon \gamma^\epsilon(\bar{Y}_0^\epsilon,\bar{Y}_1^\epsilon) = \gamma(\bar{Y}_0,\bar{Y}_1).$$

Remark. The (L^*,U^*) barrier control can be approximated for $X^\epsilon(\cdot)$ in such a way that it is of the form in (A7.1). In particular, let $\Delta_\epsilon \to 0$ as $\epsilon \to 0$ and define

$$d\tilde{Y}_1^\epsilon(t) = dt\left[F(X^\epsilon(t),\xi^\epsilon(t))/\epsilon + G(X^\epsilon(t),\xi^\epsilon(t))\right]^+ \cdot$$

(7.3)

$$\cdot\, I_{\{X^\epsilon(t)\in[U^*-\Delta_\epsilon,U^*]\}} \cdot \frac{[X^\epsilon(t) - U^* + \Delta_\epsilon]}{\Delta_\epsilon}$$

and similarly for $\tilde{Y}_0^\epsilon(\cdot)$. It can be shown that

(7.4) $$\gamma^\epsilon(\tilde{Y}_0^\epsilon,\tilde{Y}_1^\epsilon) \xrightarrow[\Delta_\epsilon\to 0]{} \gamma^\epsilon(\bar{Y}_0^\epsilon,\bar{Y}_1^\epsilon),$$

where the $\bar{Y}_i^\epsilon$ is the (L^*,U^*)-barrier policy for $X^\epsilon(\cdot)$. Clearly, the $\tilde{Y}_i^\epsilon(\cdot)$ are of the form used in (A7.1). By (7.4), for each ϵ, we can choose Δ_ϵ so that the left and right sides of (7.4) are as close as desired. By using techniques of Section 5, it can be shown that $\{\tilde{Y}_i^\epsilon(n+1) - \tilde{Y}_i^\epsilon(n),\ i = 0,1,\ n < \infty,\ \epsilon > 0,\ X^\epsilon(0) = x,\ \xi^\epsilon(0) = \xi\}$ is uniformly integrable.

Proof. For each ϵ, δ, T, define the measure

$$P_T^{\epsilon,\delta}(\cdot) = \tfrac{1}{T}\, E \int_0^T P^{\epsilon,\delta}(X^\epsilon(0),\xi^\epsilon(0),t,\cdot)dt,$$

where $P^{\epsilon,\delta}$ is the transition function for $(X^\epsilon(\cdot),\xi^\epsilon(\cdot))$, under the δ-optimal control $Y_i^\epsilon(\cdot)$ (or Q_i^ϵ) of (A7.1). Then

$$\gamma^\epsilon(Y_0^\epsilon,Y_1^\epsilon) = \overline{\lim_T} \int P_T^{\epsilon,\delta}(dxd\xi)\,[k_0\, Q_0^\epsilon(x,\xi) - k_1 Q_1^\epsilon(x,\xi) + k(x)].$$

Choose a subsequence $T \to \infty$ such that both the $\overline{\lim}$ is attained and $P_T^{\epsilon,\delta}$ converges weakly (with limit denoted by $\mu^{\epsilon,\delta}$). Then, by the Markov-Feller property of $(X^\epsilon(\cdot),\ \xi^\epsilon(\cdot))$ for $(Y_0^\epsilon(\cdot),\ Y_1^\epsilon(\cdot)) \in PM^\epsilon$, $\mu^{\epsilon,\delta}$ is an invariant measure for $(X^\epsilon(\cdot),\xi^\epsilon(\cdot))$ and (by the continuity of the Q_i^ϵ and the weak convergence,

(7.4) $$\gamma^\epsilon(Y_0^\epsilon,Y_1^\epsilon) = \int \mu^{\epsilon,\delta}(dxd\xi)\left[k_0 Q_0^\epsilon(x,\xi) - k_1 Q_1^\epsilon(x,\xi) + k(x)\right].$$

Let $(\hat{X}^\epsilon(\cdot),\hat{\xi}^\epsilon(\cdot))$ denote the *stationary process* associated with the controls $Q_i^\epsilon(\cdot,\cdot)$ and measure $\mu^{\epsilon,\delta}(\cdot)$ and let $\hat{Y}_i^\epsilon(\cdot)$ denote the corresponding stationary control processes. Then we can write (7.4) as:

$$\begin{aligned}\gamma^\epsilon(Y_0^\epsilon,Y_1^\epsilon) &= E\int_0^1 dt\left[k_0Q_0^\epsilon(\hat{X}^\epsilon(t),\hat{\xi}^\epsilon(t)) - k_1Q_0^\epsilon(\hat{X}^\epsilon(t),\hat{\xi}^\epsilon(t)) + k(\hat{X}^\epsilon(t))\right].\\ &= E\int_0^1 k(\hat{X}^\epsilon(t))dt + Ek_0\hat{Y}_0^\epsilon(1) - Ek_1\hat{Y}_1^\epsilon(1).\end{aligned} \tag{7.5}$$

By the uniform integrability (7.2), $\{\hat{Y}_0^\epsilon(1),\ \hat{Y}_1^\epsilon(1),\ \epsilon > 0\}$ is uniformly integrable.

Now, choose a weakly convergent subsequence of $\{\hat{X}^\epsilon(\cdot),Y_0^\epsilon(\cdot),Y_1^\epsilon(\cdot)\}$, with limit denoted by $(\hat{X}(\cdot),\hat{Y}_0(\cdot),\hat{Y}_1(\cdot))$. The limit is *stationary*, satisfies (1.1) and (indexing the subsequence by ϵ also), we have

$$\gamma^\epsilon(Y_0^\epsilon,Y_1^\epsilon) \to \gamma(Y_0,Y_1) \geq \gamma(\bar{Y}_0,\bar{Y}_1),$$

where the optimality of $\bar{Y}_0(\cdot)$, $\bar{Y}_1(\cdot)$ is used.

The proof is concluded by applying the same procedure to $\tilde{Y}_0^\epsilon(\cdot)$, $\tilde{Y}_1^\epsilon(\cdot)$, where the 'smoothing interval' Δ_ϵ (see remark above the theorem where $\tilde{Y}_i^\epsilon$ is defined) goes to zero fast enough as $\epsilon \to 0$.

Q.E.D.

8. The Vector Case. Formulation and Quasimartingale Estimates and the Approximation Theorem

Most of the foregoing analysis and results can be carried over to the case of vector ($x \in R^r$, Euclidean r-space) valued G, F in (1.4) or (3.1). Since the details of the proofs are essentially the same as in the foregoing sections, only an outline will be given. Only the continuous parameter case will be discussed, but under the obvious changes in the assumptions (A3.1 to A3.4) and (A8.1) used below, the discrete parameter results also extend to the vector case. Applications to queueing and production networks require a somewhat more special development, and this will be published spearately.

We use the model (vector F, G)

$$dX^\epsilon = [G(X^\epsilon,\xi^\epsilon) + F(X^\epsilon,\xi^\epsilon)/\epsilon]dt + dY^\epsilon(t), \tag{8.1}$$

with cost

$$V_0^\epsilon(x, Y^\epsilon) = \int_0^\infty e^{-\beta t}[k_0 \mid dY^\epsilon(t)| + k(X^\epsilon(t))dt], \ \beta > 0, \tag{8.2}$$

The results of Section 7 can also be extended to the vector case.

Theorem 8.1. *Assume* (A3.1, A3.2) *with vector* G, F *used, and let* $\sup_\epsilon E\int_0^T|dY^\epsilon(t) < \infty$ *for each* $T < \infty$. *Then* (*with the addition of a process whose maximum value goes to zero as* $\epsilon \to 0$) $\{X^\epsilon(\cdot), Z^\epsilon(\cdot), Y^\epsilon(\cdot)\}$ *are quasimartingales with uniformly* (*in* ϵ) *bounded variation on each bounded time interval.*

Remark. The proof is essentially identical to that of Theorem 3.1. Similarly, the proof of Theorem 8.2 below is essentially identical to that of Theorem 3.2.

We will next use (A8.1), the vector form of (A3.5).

A8.1. *There is a matrix* $\Sigma(\cdot)$ *with a continuous and bounded square root* $\sigma(\cdot)$ *such that for each* x,

$$\frac{1}{T_1}\int_u^{u+T_1} E_u F(x,\xi(\tau))d\tau \int_\tau^{T+\tau} F'(x,\xi(s))ds + \frac{1}{T_1}\left[\int_u^{u+T_1} E_u F(x,\xi(\tau))\,d\tau \int_\tau^{T+\tau} F'(x,\xi(s))ds\right]' \xrightarrow{P} \Sigma(x),$$

as T, u *and* T_1 *go to* ∞. *There is a continuous* $\bar{G}_0(\cdot)$ *with components* $\bar{G}_{0i}(\cdot)$, $i \leq r$, *such that for each* x,

$$\frac{1}{T_1}\int_u^{u+T_1} \sum_j E_u F_j(x,\xi(\tau))d\tau \int_\tau^{T+\tau} F_{ix_j}(x,\xi(s))ds \xrightarrow{P} \bar{G}_{0i}(x)$$

as T, u *and* T_1 *go to* ∞.

Theorem 8.2. *Assume* (A3.1) *to* (A3.4), *and* (A8.1) *and let* $\sup_\epsilon E_x\int_0^T |dY^\epsilon(s)| < \infty$ *for each* $T < \infty$. *Then* $\{Z^\epsilon(\cdot)\}$ *is tight in the Skorohod topology on* $D^r[0,\infty)$ *and any weak limit process is continuous* w.p.1. *Let* $\{X^\epsilon(\cdot),Y^\epsilon(\cdot),Z^\epsilon(\cdot)\}$ *be a weakly convergent subsequence in* $D^{3r}[0,\infty)$, *with the pseudopath topology used on the first two components and the Skorohod topology on the last. Let* $\bar{X}(\cdot) = (X(\cdot),Y(\cdot),Z(\cdot))$ *denote the limit of a weakly convergent subsequence. Then the conclusions of Theorem* 3.2 *continue to hold, with the limit* $Y(\cdot)$ *replacing* $Y_0(\cdot) - Y_1(\cdot)$. *In particular, the limit satisfies*

$$X(t) = Z(t) + Y(t) + X(0),\ dZ(t) = [\bar{G}(x) + \bar{G}_0(x)]dt + \sigma(x)dw, \tag{8.3}$$

$$Z(0) = 0.$$

Definition and Assumptions. Below, $v(\cdot)$ will be a continuous vector field on R^r with $|v(x)| \equiv 1$, and S a compact set with a piecewise differential boundary and with the following property: There is a $\Delta_0 > 0$ such that for $x \in \partial S$ and $\Delta \leq \Delta_0$, the points $x + \Delta v(x)$ are interior to S. Define the $(S, \Delta, v(\cdot))$-reflecting policy for

$X^\epsilon(\cdot)$ as the (admissible) policy which sets $X^\epsilon(t) = x + \Delta v(x)$, if $X^\epsilon(t^-) = x \in \partial S$. Then, of course, $dY^\epsilon(t) = \Delta v(x)$. The same definition is used for the $(S, \Delta, v(\cdot))$-reflecting policy for the $X(\cdot)$ of (8.3).

A policy $Y(\cdot)$ for (8.3) is called a $(S, v(\cdot))$-reflecting policy if the associated process $X(\cdot)$ is a reflected diffusion in S, with continuous reflection direction $v(\cdot)$ on ∂S, and there is a $\Delta_0 > 0$ such that for $\Delta \leq \Delta_0$, the policy which sets $X(t) = x + \Delta v(x)$ if $X(t^-) = x \in \partial S$ is an admissible $(S, \Delta, v(\cdot))$-reflecting policy.

Theorems 5.1 and 5.2 continue to hold. Here, we would require that the Q_ϵ of these theorems be such that it guarantees boundedness of the $X^\epsilon(\cdot)$; e.g., choose a bounded set S_1, and let Q_ϵ simply just push 'enough' to keep $X^\epsilon(\cdot)$ from leaving that set. In the approximation Theorem 8.5, we use a different approach, based on the use of a $(S,\Delta,v(\cdot))$-reflecting policy to approximate a 'reflecting' diffusion. We will use only the following two theorems.

Theorem 8.3. *Assume* (A3.1) *and* (A3.2) (*vector case*), *and let* $Y^{\epsilon,\Delta}(\cdot)$ *denote a* $(S,\Delta,v(\cdot))$-*reflecting policy for* $X^\epsilon(\cdot)$. *Then for each* $T < \infty$ *and integer* k

$$\sup_{\substack{\epsilon,\Delta\leq\Delta_0 \\ x\in S}} E_x\left[\int_0^T |dY^{\epsilon,\Delta}(s))|\right]^k < \infty . \tag{8.2}$$

Theorem 8.4. *Assume the model* (8.3) *with bounded and continuous* $\bar{G}(\cdot)$, $\bar{G}_0(\cdot)$ *and* $\sigma(\cdot)$. *Let* $Y^\Delta(\cdot)$ *be a* $(S,\Delta,v(\cdot))$-*reflecting policy for* $x(\cdot)$. *Then, for each* $T < \infty$ *and integer* k,

$$\sup_{\substack{\Delta\leq\Delta_0 \\ x\in S}} E_x\left[\int_0^T |dY^\Delta(s)|\right] < \infty .$$

We will prove Theorem 8.4 only. The proof of Theorem 8.3 is similar, and uses the (vector case) estimate (5.6) for the process $Z^\epsilon(t) + Z_1^\epsilon(t)$, where $Z_1^\epsilon(\cdot)$ is the appropriate 'vector' case replacement for the $Z_1^\epsilon(\cdot)$ used in Sections 3 to 5.

Proof of Theorem 8.4. Let $Y^\Delta(\cdot)$ denote the $(S,\Delta,v(\cdot))$-reflecting policy and $X^\Delta(\cdot)$ the associated solution to (8.3). Fix $\alpha > 0$ and small. Let $N_\alpha(x)$ denote the α-neighborhood of x. There are $x_1, ..., x_q$ on ∂S such that $\cup_{i=1}^1 N_\alpha(x_i) \supset \partial S$ and

$$\sup_{x,y \in N_{2\alpha}(x_i)} |v(x) - v(y)| < \alpha. \tag{8.5}$$

Let σ_0^m denote the first time of entry of $X^\Delta(\cdot)$ into $\bar{N}_\alpha(x_m)$. Define

$$\rho_i^m = \min\{t > \sigma_{i-1}^m : X^\Delta(t) \notin N_{2\alpha}(x_m)\}$$
$$\sigma_i^m = \min\{t > \rho_i^m : X^\Delta(t) \in \bar{N}_\alpha(x_m)\}.$$

Define $N_m^\Delta = \max\{i: \sigma_i^m \leq T\}$, $Y_i^{\Delta,m}(T) = Y^\Delta(\rho_{i+1}^m \cap T) - Y^\Delta(\sigma_i^m \cap T)$ and $Y^{\Delta,m}(T) = \Sigma_i Y_i^{\Delta,m}(T)$.

Owing to (8.5) and the smallness of α, there is a $K_1 < \infty$ (depending on α, but not on Δ) such that

$$\int_0^T |dY^\Delta(s)| \leq K_1 \sum_{m=1}^q |Y^{\Delta m}(T)|.$$

Hence we need only evaluate $E[Y^{\Delta,m}(T)]^k$. We have

$$\begin{aligned} Y^{\Delta,m} &= \sum_1^{N_m^\Delta} [X^\Delta(\rho_i^m \cap T) - X^\Delta(\sigma_{i-1}^m \cap T)] \\ &\quad - \sum_{i=1}^{N_m^\Delta} [Z^\Delta(\rho_i^m \cap T) - Z^\Delta(\sigma_{i-1}^m \cap T)]. \end{aligned} \tag{8.6}$$

The absolute value of the first term on the r.h.s. of (8.6) is $\leq \alpha K_2 N_m^\Delta$, for some constant $K_2 < \infty$. The absolute value of the last term on the r.h.s. of (8.6) is bounded above by

$$N_m^\Delta \cdot \sup_{s,t \leq T} |Z^\Delta(t) - Z^\Delta(s)|. \tag{8.7}$$

Since $z^\Delta(\cdot)$ is just the sum of an ordinary integral and a stochastic integral whose integrands are bounded uniformly in Δ, all the moments of the last factor in

(8.7) are bounded uniformly in Δ. Hence it is enough to show that $\sup_{x\in S,\Delta} E_x|N_m^\Delta|^p < \infty$ for all integers p.

This last problem is similar to that dealt with in Theorem 5.3. Owing to the nature of the $(S,\Delta,v(\cdot))$ - reflecting policy there is an $\alpha' > 0$ (not depending on Δ) such that in order for $X^\Delta(\cdot)$ to move from the exterior of $N_{2\alpha}(x_m)$ at time ρ_i^m to $\bar{N}_\alpha(x_m)$ at time σ_i^m, we must have $\sup_{t\le\sigma_i^m} |Z^\Delta(t) - Z^\Delta(\rho_i^m)| \ge \alpha'$. Let τ be a finite stopping time. For each $\delta_0 \in (0,1)$, there is a $T_0 > 0$ such that for all small $\Delta > 0$,

$$\sup_{x\in S,\tau} P_x\left\{\sup_{t\le T_0} |Z^\Delta(t+\tau) - Z^\Delta(\tau)| \ge \alpha'/2\right\} \le 1 - \delta_0 . \tag{8.8}$$

The inequality (8.8) and an argument like that used in Theorem 5.3 (to get the upper estimate on $E|N_i^\epsilon|^k$ there) completes our proof.

Q.E.D.

Definitions. We now add an additional qualification on the control problem. It is supposed that there is a compact set S_1 with a piecewise differentiable boundary such that $X^\epsilon(\cdot)$ and $X(\cdot)$ are to be confined to S_1. Let there exist a $(S_1,\Delta,v_1(\cdot))$-reflecting policy for small Δ and some continuous $v_1(\cdot)$. As noted in the remarks after the theorem, the approximation theorem is easier to prove without this restriction. In the theorem, we assume that the optimal control for $X(\cdot)$ is $(S,v(\cdot))$-reflecting for some S (compact, since S_1 is compact). This will often be the case. But, as noted in the introduction, other forms are possible: combined singular and non-singular controls, true impulsive controls, etc.

Theorem 8.5. *Assume* (A3.1) - (A3.4) *and* (A8.1), *and the condition in the above paragraph. Suppose that the optimal policy* $\bar{Y}(\cdot)$ *for* $X(\cdot)$ *is a* $(S,v(\cdot))$*-reflecting policy for some bounded* S. *Let* $\bar{Y}^\Delta(\cdot)$ *denote its* $(S,\Delta,v(\cdot))$ *reflecting form. Let* (8.3) *have a unique weak sense solution under both policies, for all small* Δ. *Let* $\hat{Y}^{\epsilon,\Delta}(\cdot)$ *denote the*

$(S,\Delta,v(\cdot))$-*reflecting policy adapted to* $X^\epsilon(\cdot)$. *Given* $\delta > 0$, *there is a* $\Delta > 0$ *such that* $\hat{Y}^{\epsilon,\Delta}(\cdot)$ *is* 2δ-*optimal for* $X^\epsilon(\cdot)$ *and small* ϵ *in the sense that*

$$(8.9) \qquad \delta + \underline{\lim}_\epsilon V^\epsilon(x) \geq \lim_\epsilon V_0^\epsilon(x, \hat{Y}^{\epsilon,\Delta}) = V_0(x, \bar{Y}^\Delta),\ V_0(x,\bar{Y}^\Delta) \leq V(x) + \delta.$$

Proof. The method is that of Theorems 6.1 and 6.2. Let $Y^\epsilon(\cdot)$ denote the optimal (or $\delta/4$-optimal, if there is no optimal policy) policy for $X^\epsilon(\cdot)$ and $X^{\epsilon,\Delta}(\cdot)$ the process corresponding to $\hat{Y}^{\epsilon,\Delta}(\cdot)$. By the argument of Theorem 6.2, there is no loss of generality if we suppose that the second set of

$$\left\{\int_n^{n+1} |d\hat{Y}^{\epsilon,\Delta}(s)|,\ \epsilon > 0,\ n < \infty\right\},\ \left\{\int_n^{n+1} |dY^\epsilon(s)|,\ \epsilon > 0,\ n < \infty\right\}$$

is uniformly integrable. The first set is uniformly integrable by Theorem 8.3. Let ϵ index a weakly convergent subsequence $\{X^{\epsilon,\Delta}(\cdot),\ \hat{Y}^{\epsilon,\Delta}(\cdot)\}$ and $\{X^\epsilon(\cdot),\ Y^\epsilon(\cdot)\}$ with limit pairs $(X^\Delta(\cdot),\ \hat{Y}^\Delta(\cdot))$ and $(X(\cdot),\ Y(\cdot))$. Then $X^\Delta(\cdot)$ is the $(S,\Delta,v(\cdot))$ - reflecting diffusion and $\hat{Y}^\Delta(\cdot) = \bar{Y}^\Delta(\cdot)$. Thus, by the weak convergence

$$\delta/4 + \underline{\lim}_\epsilon V^\epsilon(x) \geq \underline{\lim}_\epsilon V_0^\epsilon(x, Y^\epsilon) \geq V_0(x, Y) \geq V_0(x, \bar{Y}) = V(x),$$

$$\lim_\epsilon V^\epsilon(x,\hat{Y}^{\epsilon,\Delta}) = V(x,\bar{Y}^\Delta).$$

Another weak convergence argument and the uniqueness assumption on the reflecting diffusion $X(\cdot)$ under policy $\bar{Y}(\cdot)$ yields the convergence of $\{X^\Delta(\cdot),\bar{Y}^\Delta(\cdot)\}$ to $(X(\cdot),\ \bar{Y}(\cdot))$. Also, the set $\{\int_n^{n+1}|d\bar{Y}^\Delta(s)|,\ \Delta \leq \Delta_0,\ n < \infty\}$ is uniformly integrable. The theorem now follows by choosing Δ small enough so that $\bar{Y}^\Delta(\cdot)$ is $\delta/4$ optimal for $X(\cdot)$ and using the optimality of $\bar{Y}(\cdot)$ for $X(\cdot)$.

Q.E.D.

Remarks and Extensions. If the bounding set S_1 is dropped, then we might assume that the optimal control is a $(S,v(\cdot))$-reflecting policy, but where S is not necessarily compact. In this case, given $\delta > 0$, there are numbers K_δ and $\delta/4$- optimal policies for $X^\epsilon(\cdot)$ and $X(\cdot)$ for which $dY^\epsilon(t)$ (resp., $dY(t)$) equals zero after the first time that the variation exceeds K_δ. In this case, we have the required uniform integrability and the theorem is easier to prove.

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